FREE VIBRATION ANALYSIS OF ROTATIONALLY PERIODIC STRUCTURES USING METHOD OF COMPLEX CONSTRAINTS

by

AMUL VERMA

DEPARTMENT OF MECHANICAL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY KANPUR

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CERTIFICATE

This is to certify that this work on "Free Vibration Analysis Rotationally Periodic Structures Using Method of Compl Constraints", by Amul Verma has been carried out under o supervision and that it has not been submitted elsewhere for degree.

Dr. P. M. Dixit
(Assistant Professor)
Dept. of Mechanical Engg.
I. I. T. Kanpur

Dr. B. L. Dhoopar
(Professor)
Dept. of Mechanical I
I. I. T. Kanpur

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CONTENTS

LIST OF SYMBOLS

$[k]$	element stiffness matrix
$[m]$	element mass matrix
$[k_a]$	element stiffness matrix for longitudinal vibrations
$[m_a]$	element mass matrix for longitudinal vibrations
$[k_b]$	element stiffness matrix for transverse vibrations
$[m_b]$	element mass matrix for transverse vibrations
$[k]^{(et)}$	element stiffness matrix after applying transformation
$[m]^{(et)}$	element mass matrix after applying transformation
$[T]$	Transformation matrix
$[K]_{ij}^{(i)}$	stiffness submatrices of i^{th} substructure
$[M]_{ij}^{(i)}$	mass submatrices of i^{th} substructure
$[K]$	Global stiffness matrix
$[M]$	Global mass matrix
$\{u\}$	eigen vector representing the mode shape of rotationally periodic structure
$\{\bar{u}\}$	eigen vector orthogonal to $\{u\}$
$\{u'\}$	eigen vector obtained by rotating $\{u\}$ through one substructure
$\{\bar{u}'\}$	eigen vector orthogonal to $\{u'\}$
$\{u\}^{(j)}$	eigen vector for j^{th} substructure
$\{Z\}$	complex eigen vector

$\{Z\}^{(j)}$	complex eigen vector for j^{th} substructure
$\{f\}$	nodal force vector
E	Young's modulus of elasticity
A	area of cross section
I	area moment of inertia about an axis normal to both axial and transverse direction
h	length of the finite element
J	number of degrees of freedom
N	number of substructures
n	an integer less than N
u	longitudinal displacement
v	transverse displacement
Ni's	shape functions
S	shear force
M	bending moment
t	time
ψ	phase angle between substructures
ω	natural frequency
θ	first partial derivative of displacement with respect to space coordinates

LIST OF FIGURES

LIST OF TABLES

ABSTRACT

In the present work, finite element analysis of free vibrations of two rotationally periodic structures, namely, banded blades and finite continuous beam, has been considered, using the method of complex constraints.

Mode shapes of rotationally periodic structures can be classified into the classes of symmetric modes, anti-symmetric modes and the rest. In the third category, the eigenvalues have multiplicity of two, i.e., corresponding to every natural frequency, there exist an orthogonal pair of mode shapes with eigenvector $\{u\}$ and $\{\bar{u}\}$. The complex eigenvector $\{Z\} = \{u\}+i\{\bar{u}\}$ is also an eigenvector of equation of motion. It is observed that, the deflection of one substructure has the same amplitude and a constant phase difference from the preceeding substructure. It is therefore possible to express the mass and stiffness matrices of complete structure, in terms of the first substructure only. Therefore a complete structure can be analysed by considering only the first substructure. This is called method of complex constraints. In this method, the elements of stiffness and mass matrices are complex numbers. A computer programme has been made to implement the method and it has been illustrated with the help of two problems as mentioned above. IMSL subroutine DGVCCG has been used to find eigenvalues and associated eigenvectors of the complex generalized eigenvalue problem.

For the analysis of banded blades both axial and transverse vibrations have been considered, whereas for continuous beam only transverse vibrations have been considered. Results have been compared with the natural frequencies and mode shapes, obtained by standard finite element method. It is observed that, the present method gives quite accurate results, and computer memory requirements as well as computation time are much less as compared to conventional finite element method.

CHAPTER I

INTRODUCTION

1.1 GENERAL INTRODUCTION

There are many instances in engineering wherein, the structures encountered are periodic in nature. The most common example is a continuous beam with equal span length. Besides, there are many structures which can be modelled as a continuous beam. Usually a simple model of a common aircraft fuselage is a beam on periodically spaced supports. Structures usually have two kinds of periodicities viz. linear and cyclic or rotational periodicity. In a periodic structure a single basic unit (substructure) repeats itself within the structure. In a linear periodic structure the substructure repeats along a line. A cyclic periodic structure is a special case of periodic structures in which the starting point of the first substructure is the same as the last point of the last substructure. Linear and cyclic periodicity is discussed in more detail in subsequent chapters. Besides a continuous beam, another common example of a linear periodic structure is a plane truss.

Examples of cyclic periodic structures occurring in nature are numerous. A simple model of banded blades of turbine discs and rotors of pumps and compressors, is also a cyclic periodic structure. Cooling towers with legs used in power plants is also

a cyclic periodic structure. All types of gears and rings can also be modelled as cyclic periodic structures, and in electrical industries the alternator end windings are also cyclic periodic in nature.

The structures, mentioned so far, have periodicity properties in one direction only. There are some structures, which have periodicity properties in two directions. They are called doubly periodic structures. Grillage used for roofing, the ruber tube of a bicycle, are some of the existing doubly periodic structures.

1.2 LITERATURE SURVEY

The natural vibrations of various periodic structures have been studied by several authors. D. J. Mead [1] used wave propagation method to study the natural vibrations of infinite continuous beam. He described the phenomenon of wave propagation by a propagation constant. G. Sen Gupta [2] extended it to the case of finite continuous beam with various end conditions. They could reduce the free vibration analysis of complete structure to that of a substructure only, but their method is applicable only to the structures of simple geometry. B. L. Dhoopar, P. C. Gupta and B. P. Singh [3] used transfer matrix method to analyze vibrations of cable networks. They obtained the transfer matrix for individual members, from the closed form solution of the equations of motions. These transfer matrices were then used to derive the transfer matrix for the whole system.

The above mentioned methods have been employed for vibrations analysis of linear periodic structures. In 1979, D. L. Thomas [4] reduced the finite element analysis of vibrations of cyclic periodic structures, to that of one substructure only, by the method of complex constraints. C. W. Cai, Y. K. Cheung and H. C. Chan [5] proposed a 'U' transformation method for cyclic periodic structures. They expressed the response of a system in terms of a series of mode subspaces, multiplied by a transformation matrix, which they called the 'U' transformation matrix. For doubly cyclic periodic structures double 'U' transformation is required. The authors have employed 'U' transformation method to study the transverse vibrations of plane trusses[6].

1.3 SCOPE OF THE PRESENT WORK

In the present work free vibration analysis of two cyclic periodic structures have been carried out. The first problem is the study of free vibrations of banded blades and the second problem deals with two span beam with various end conditions. Both the problems have been solved by two methods. The first method used is the method of complex constraints, proposed by D. L. Thomas. In the second method finite element assembly for complete structure is done to obtain the natural frequencies and associated mode shapes.

1.3.1 Problem 1 : Banded blades

The first problem analysed is a linear structure formed by

banded blades as shown in fig (3.1a). This structure can be used to model a specific stage of a turbine, with banded blades. If the radius of the turbine disc is made very large and the disc is considered to be rigid, then the structure can be idealized as a linear structure with fixed supports.

The problem has been solved by the method of complex constraints as well as by the standard finite element method. Results are discussed in chapter III.

1.3.2 Problem 2 :Finite continuous beam with three different end conditions

The second problem deals with, natural frequencies and mode shapes of a two span beam with the following three sets of end conditions:

i) Both extreme ends hinged,

ii) Both extreme ends clamped,

iii) One end clamped, other hinged.

For all of the above mentioned cases the beam does not satisfy the condition for cyclic periodicity. So the equivalent cyclic periodic structure is proposed, which takes care of all the three types of end conditions. It is shown in chapter III that for the above two-span beam the equivalent cyclic periodic structure is an eight span continuous beam resting on hinge supports. In this case also the problem has been solved by the method of complex constraints, And the results have been compared with the results obtained by finite element assembly for the

complete structure i.e. the two span beam.

In both the cases the continuum has been discretized by beam elements. The finite element assembly for complete structure and subsequent application of boundary condition results in the following generalized eigenvalue equation,

$$([K] - \omega^2 [M]) \{u\} = \{0\} \qquad (1.1)$$

where [K] and [M] are stiffness and mass matrices respectively and {u} is the displacement vector. Solution of equation (1.1) gives eigenvalues ω^2 and associated eigenvectors {u}.

The method of complex constraints reduces equation (1.1) to that involving stiffness and mass matrices of first substructure only, with stiffness and mass matrices having its elements as complex numbers. The method directly gives orthogonal pairs of eigenvectors, for multiple eigenvalues.

In chapter II, a brief discussion on periodic structure is given, followed by the finite element formulation for natural vibration of structures. Finally in this chapter the method of complex constraints has been detailed out. The discussion and the analysis of results is presented in chapter III. Chapter IV summarizes the conclusions and suggestions for future work.

CHAPTER II

THEORY AND PROBLEM FORMULATION

In engineering practice, there are many structures encountered, which are periodic in nature. Rings, banded blades of turbine discs , fuselage of an aircraft, rotors of pumps and compressors are some of the commonly occurring periodic structures.

2.1 PERIODIC STRUCTURES

By periodic, it is generally meant that, something appearing at regular interval of time or over some interval of length in space. In mathematics, a periodic function is defined as, a function $f(x)$, such that, it is defined for all x and there exists a $P > 0$, such that:

$$f(x+P) = f(x).$$

P is called the period of the function and the function repeats after intervals of P. Similarly a periodic structure can be defined as a structure, in which, the geometry, defined over some length of the domain, repeats over the entire domain. A periodic structure can thus be divided in to a number of *identical* substructures. Once the substructure has been defined,

the whole structure can be generated by repetition of the substructure.

2.1.1 Linear periodic structure

In a linear periodic structure, the substructure repeats itself along a line in space. Continuous beam and plane truss are linear periodic structures.

2.1.2 Rotationally periodic structures

A rotationally or cyclic periodic structure is one, in which the starting point of the first substructure is same as the last point of the last substructure.

The projection of the centre line of the geometry of a cyclic periodic structure, in a plane perpendicular to it's axis, forms either a circle or a polygon. If the geometry has been defined at some angle θ, then it is identical at $(\theta+n\psi)$, where ψ is $2\pi/N$, and n & N are integers. N is the number of repeating units (substructure), and n is any integer less than N. It follows that, once the geometry has been defined over a sector from θ to $(\theta+\psi)$, the remainder of the structure can be obtained by repeated rotations through ψ. Rings, banded blades of turbine discs, rotors of pumps and compressors, all are rotationally periodic structures.

Some of the linear periodic structures can also be modelled as rotationally periodic structures, by imposing certain constraints at the two extreme ends, such that these two ends mathematically represent the same point. These constraints are equality of slopes, displacements and bending moments at the two extreme ends.

2.1.3 Doubly periodic structures

Structures having periodicity properties in two directions are called doubly periodic structures. A doubly periodic structure may possess linear or cyclic periodicity in two directions or it may have a combination of the two.

Doubly cyclic periodic structures are not very common. However, an infinitely long cylindrical shell with periodic stiffening by ribs in axial and circular directions, is a doubly cyclic periodic structure. Similarly rubber tube of a bicycle, which has a torous shape, is also a doubly cyclic periodic structure. Grillage used for roofing, is another example of a doubly periodic structure. This has linear periodicity in two directions. As mentioned earlier such structures can be analysed as cyclic periodic structure by imposing constraints at the ends.

2.2 FEM FORMULATION OF SIMPLE STRUCTURES

Structures considered in the present work consist of straight bars. Vibrations of straight bars are governed by a second order partial differential equation for longitudinal vibrations, and by a fourth order partial differential equation for transverse vibrations. Structures have infinite degrees of freedom and a closed form solution can be found only for structures having simple geometry. For finding solutions in the case of complex geometries approximate methods are used. Finite element method is the most powerful among the existing approximate methods.

In the finite element analysis of free vibrations of a structure, the continuum which has infinite degrees of freedom is discretized. Discretization converts it into a finite degrees of freedom system. The element stiffness and mass matrices are then obtained. Finally, these element stiffness and mass matrices are assembled in the global ones. The partial differential equation thus reduces to a set of simultaneous algebraic equations, which in matrix form can be written as,

$$([K] - \omega^2 [M]) \{u\} = \{0\} \qquad (2.1)$$

where, [K] and [M] are global stiffness and mass matrices respectively. The solution of the above equation gives natural

frequencies ω and eigen vectors $\{u\}$.

To obtain global stiffness and mass matrices, the element stiffness and mass matrices are first found. In the present work structures have been discretized by beam elements. Both, longitudinal and transverse vibrations have been considered. Here, the analysis has been carried out for one element, which is true for all the elements.

2.2.1 Longitudinal vibrations

The equation of motion, for longitudinal vibrations of bars is

$$\rho A\ \partial^2 u/\partial x^2 - EA\ \partial^2 u/\partial t^2 = 0 \qquad (2.2)$$

where, ρ is the mass density of the material of the beam, A is area of cross section and E is Young's modulus of elasticity.

Assuming linear variation for axial displacement u,

$$u^{(e)} = \{ N_1 \quad N_2 \} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad (2.3)$$

where u_1 and u_2 are node displacements and N_1 and N_2 are shape functions given as,

$$\left.\begin{aligned} N_1 &= 1 - x/h \\ N_2 &= x/h \end{aligned}\right\} \qquad (2.4)$$

where, h is the length of the element,

Substituting displacement function (2.3) in equation(2.2) and applying Galerkin's method, the following equation is obtained:

$$\int Ni \left[\partial/\partial x \left(EA\, \partial u^{(e)}/\partial x\right) - \rho A\, \partial^2 u^{(e)}/\partial t^2 \right] dx = 0$$

Integration by parts and further simplification leads to,

$$[m] \langle \ddot{u} \rangle^{(ne)} + [k] \langle u \rangle^{(ne)} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad (2.5)$$

where, the nodal forces are given by,

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} -EA\, \partial u/\partial x \,\big|_{x=0} \\ EA\, \partial u/\partial x \,\big|_{x=h} \end{Bmatrix}$$

and element stiffness and mass matrices are given by the following equation:

$$[k] = \int EA \langle N' \rangle \langle N' \rangle^T dx$$

$$[m] = \int \rho A \langle N \rangle \langle N \rangle^T dx \qquad (2.6)$$

On substituting the expressions for shape functions and integrating equation (2.6), the stiffness and mass matrices come out to be,

$$[m] = \rho Ah/6 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad (2.7a)$$

$$[k] = EA/h \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad (2.7b)$$

2.2.2 Transverse vibrations of beams

The equation of motion for transverse vibration of beams is,

$$EI\ \partial^4 v/\partial x^4 + \rho A\ \partial^2 v/\partial t^2 = 0 \qquad (2.8)$$

where, I is the area moment of inertia of cross section of beam, about an axis perpendicular to both axial and transverse directions.

For a beam element, every node has two degrees of freedom viz. v and v'.

The displacement function assumed is,

$$v^{(e)} = N_1\ v_1 + N_2\ v_1' + N_3\ v_2 + N_4\ v_2' \qquad (2.9)$$

where v_1, v_1' and v_2, v_2' are transverse displacements and slopes at node 1 and node 2 respectively.

Shape functions N_1 , N_2 , N_3 , and N_4 are given by

$$N_1 = 1 - 3\ x^2/h^2 + 2\ x^3/h^3$$

$$N_2 = x - 2\ x^2/h + x^3/h^2$$

$$N_3 = 3\ x^2/h^2 - 2\ x^3/h^3$$

$$N_4 = -\ x^2/h^2 + x^3/h^3 \qquad (2.10)$$

Substituting the assumed displacement function (2.9) in

equation of motion (2.8) and applying Galerkin's method the following equation is obtained,

$$\int N_i \, [\partial^2/\partial x^2 \, (EI \, \partial^2 v^{(e)}/\partial x^2) + \rho A \, \partial^2 v^{(e)}/\partial t^2] \, dx = 0$$

Integration by parts and further simplification results in,

$$[m] \{ \ddot{v} \}^{(ne)} + [k] \{v\}^{(ne)} = \{ f \}^{(ne)} \qquad (2.11)$$

where $\{f\}^{(ne)}$, the nodal force vector is given by,

$$\{f\}^{(ne)} = \{ -S_1 \quad -M_1 \quad S_2 \quad M_2 \}^T$$

where, S_1, M_1 and S_2, M_2 are shear force and bending moment at node 1 and node 2 respectively. The element stiffness and mass matrices are given by,

$$[m] = \int \rho A \, \{N\} \, \{N\}^T \, dx, \quad \text{and}$$

$$[k] = \int EI \, \{N''\} \, \{N''\}^T \, dx \qquad (2.12)$$

On substituting the expressions for shape functions (2.10) and integrating equations (2.12) the stiffness and mass matrices obtained are the following,

$$[m] = \rho Ah/420 \begin{bmatrix} 156 & 22h & 54 & -13h \\ 22h & 4h^2 & 13h & -3h^2 \\ 54 & 13h & 156 & -22h \\ -13h & -3h^2 & -22h & 4h^2 \end{bmatrix} \qquad (2.13)$$

$$[k] = EI/h^3 \begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix} \qquad (2.14)$$

2.2.3 Assembly

After the element stiffness and mass matrices have been set up, the next step is assembling these matrices in global ones Corresponding to a component of element stiffness and mass matrices, its location in global stiffness and mass matrices is determined using the connectivity. This is done for every element and the contributions of all elements are added.

As an example, consider the case of bending vibrations with element stiffness and mass matrices given by equations (2.13) and (2.14). After the assembly, equation (2.11) will look like,

$$[M] \{\ddot{v}\} + [K] \{v\} = \begin{Bmatrix} -S_1 \\ -M_1 \\ | \\ 0 \\ | \\ S_n \\ M_n \end{Bmatrix} \qquad (2.15)$$

In the above equation, the shear forces and bending moments at the intermediate nodes gets cancelled because of equal and

opposite contributions from the neighbouring elements. Thus in the unknowns in equation (2.15) are node displacements and shear force and bending moment terms S_1, M_1, and S_n and M_n.

Boundary Conditions

The typical sets of boundary conditions are the following.

1) One end is fixed and other free (fig 2.1a)

In this case boundary conditions will be

$v_1 = 0 \qquad S_n = 0$

$v_1' = 0 \qquad M_n = 0$

2) Both ends hinged (fig 2.1b)

Boundary conditions in this case are

$v_1 = 0 \qquad v_n = 0$

$M_1 = 0 \qquad M_n = 0$

3) One end fixed, other hinged (fig 2.1c)

In this case boundary conditions are

$v_1 = 0 \qquad v_n = 0$

$v_1' = 0 \qquad M_n = 0$

As an illustration of the application of boundary conditions, consider the first case. The first two rows and columns from equation (2.15) are removed to apply the conditions,

$v_1=0$ and $v_1'=0$. Equation (2.15) will reduces to,

$$[M]\ \{\ddot{v}\} + [K]\ \{v\} = \{0\} \qquad (2.16)$$

Now putting,

$$v=v_0\ \exp(i\omega t)$$

$$\ddot{v} = -\omega^2\ v_0\ \exp(i\omega t)$$

Equation (16) reduces to,

$$(\ [K] - \omega^2\ [M]\ )\ \{v\} = \{0\} \qquad (2.17)$$

which is same as Equation (1.1). This eigen value equation can be solved to give natural frequencies ω and eigen vectors $\{v\}$.

Equation (1.1) has been arrived at by applying boundary conditions for case 1. For any other set of end conditions also, similar equation will be obtained, except that, the size of stiffness and mass matrices will be different, depending upon the number of degrees of freedom suppressed to take care of end conditions.

The size of stiffness and mass matrices in equation (2.15) increases as the number of substructures increase. Therefore, when the no of substructures are large, solving the eigenvalue equation (2.17) requires large computer memory and computational effort.

The case of periodic structures, all the substructures are *identical*. This property of periodic structures can be

exploited to reduce the size of stiffness and mass matrices. D. L. Thomas [4] has shown that the vibration analysis of periodic structures can be reduced to that of a single substructure only. Since the present work is based on the work of D. L. Thomas [4], his theory for rotationally periodic structure is being reproduced here.

2.3 FEM FORMULATION OF ROTATIONALLY PERIODIC STRUCTURES

Consider a rotationally periodic structure consisting of N identical substructures. Let each substructure have J degrees of freedom. The total number of degrees of freedom will thus be NJ. A typical eigen vector {u} of the whole structure can be partitioned into N subvectors each having J components. The vector {u} can be written as,

$$\{u\} = \{ \{u^{(1)}\} \ \{u^{(2)}\}\ldots\ldots\ldots\{u^{(N)}\} \}^T \qquad (2.18)$$

where $\{u\}^{(j)}$ is the eigen vector associated with the j^{th} substructure. Mode shapes for rotationally periodic structures can be classified in the following three categories:

(a) In the first category are those modes, in which each substructure has the same mode shape as its neighbours i.e.,

$$u^{(j)} = u^{(j+1)} \qquad , \text{ for all } j$$

For such mode shapes an eigenvector for the complete

structure can be written as

$$\{u\} = \{ u^{(1)} \quad u^{(1)} \quad u^{(1)} \quad \ldots\ldots\ldots u^{(1)} \}^T \tag{2.19a}$$

(b) In this class, each substructure has the same mode shapes as its neighbours but is vibrating in anti-phase with them. Therefore the mode shape for the whole structure can be expressed as,

$$\{u\} = \{ u^{(1)} \quad -u^{(1)} \quad u^{(1)} \quad \ldots\ldots\ldots -u^{(1)} \}^T \tag{2.19b}$$

(c) All other possible modes fall in the third category for which,

$$u^{(j)} \neq u^{(j+1)} \qquad \text{and} \qquad u^{(j)} \neq -u^{(j+1)} \quad , \text{ for all } j$$

Class (c) modes are associated with multiple eigen values. For such cases, associated with each eigenvalue are a number of eigenvectors lying in a 2-D subspace of n-dimensional space. Any eigen vector lying in the 2-D subspace can be represented as a linear combination of two orthogonal vectors, contained in the same subspace.

2.3.1 Properties of eigenvectors

In this section class (c) eigenvectors will be considered in detail. The eigenvectors are normalized such that $\{u\} \{u\}^T = 1$. The eigenvector, obtained by rotating $\{u\}$ through one substructure, is also an eigen vector with the same eigen

value as that of $\{u\}$. Let this vector be $\{u'\}$, which can be expressed as,

$$\{u'\} = \{ \{u^{(N)}\} \quad \{u^{(1)}\} \quad \{u^{(2)}\}.....\{u^{(N-1)}\} \}^T \qquad (2.20)$$

Note that $\{u'\}$ may or may not be orthogonal to $\{u\}$. Let the vector orthogonal to $\{u\}$ be denoted by $\{\bar{u}\}$, which has the same frequency as $\{u\}$. The vector $\{\bar{u}\}$ is also normalized such that $\{\bar{u}\}^T\{\bar{u}\} = 1$. The vector $\{u'\}$ can be expressed as a linear combination of the two orthogonal eigenvectors $\{u\}$ and $\{\bar{u}\}$,

$$\{u'\} = a\{u\} + b\{\bar{u}\} \qquad (2.21)$$

where a and b are scalars.

The vector $\{u'\}$ has been obtained by rotating $\{u\}$ through one substructure. If $\{\bar{u}\}$ is rotated in the same direction, through one substructure, the vector obtained will be orthogonal to $\{u'\}$. Let this vector be denoted by $\{\bar{u}'\}$. This is associated with the same eigen value as that of $\{u'\}$. This can also be written as a linear combination of the two orthogonal vector $\{u\}$ and $\{\bar{u}\}$, in the following way,

$$\{\bar{u}'\} = -b\{u\} + a\{\bar{u}\} \qquad (2.22)$$

It is to be noted that when $\{u\}$, $\{\bar{u}\}$ and $\{u'\}$ are normalized, a and b are respectively $\cos\psi$ and $\sin\psi$. Therefore $\{\bar{u}'\}$ automatically gets normalized.

Substituting $\{u'\}$ from equation (2.20) in equation (2.21)and

then comparing the left and the right hand sides the following relations will be obtained:

$$\left.\begin{aligned} \{u^{(N)}\} &= a\,\{u^{(1)}\} + b\,\{\bar{u}^{(1)}\} \\ \{u^{(1)}\} &= a\,\{u^{(2)}\} + b\,\{\bar{u}^{(2)}\} \\ \{u^{(2)}\} &= a\,\{u^{(3)}\} + b\,\{\bar{u}^{(3)}\} \end{aligned}\right\}$$

and so on.. (2.23)

The above relations in general can be written as follows:

$$\{u^{(j-1)}\} = a\,\{u^{(j)}\} + b\,\{\bar{u}^{(j)}\} \quad ,\text{for all } j \qquad (2.24a)$$

Similarly one can write,

$$\{\bar{u}^{(j-1)}\} = -b\,\{u^{(j)}\} + a\,\{\bar{u}^{(j)}\} \quad ,\text{for all } j \qquad (2.24b)$$

2.3.2 Complex eigenvectors

The eigenvectors $\{u\}$ and $\{\bar{u}\}$ as well as $\{u'\}$ and $\{\bar{u}'\}$ are real, normalized, orthogonal pairs of eigenvectors and are associated with same eigen value. The linear combination of these vectors can be written more conveniently in the form complex eigen vectors $\{Z\}$ and $\{Z'\}$ as,

$$\begin{aligned} \{Z\} &= \{u\} + i\,\{\bar{u}\} \\ \{Z'\} &= \{u'\} + i\,\{\bar{u}'\} \end{aligned} \qquad (2.25)$$

In terms of subvectors one can write,

$$\begin{aligned} \{Z^{(j)}\} &= \{u^{(j)}\} + i\,\{\bar{u}^{(j)}\} \\ \{Z'^{(j)}\} &= \{u'^{(j)}\} + i\,\{\bar{u}'^{(j)}\} \end{aligned} \qquad (2.26)$$

from the above equation,

$$\langle Z^{(j-1)}\rangle = \langle u^{(j-1)}\rangle + i\ \langle \bar{u}^{(j-1)}\rangle$$

Substituting from equation (2.24a) and (2.24b) the expressions for $\langle u^{(j-1)}\rangle$ and $\langle \bar{u}^{(j-1)}\rangle$ and with $a=\cos\psi$ and $b=\sin\psi$, equation (2.26) reduces to,

$$\langle Z^{(j-1)}\rangle = e^{-i\psi}\ \langle Z^{(j)}\rangle$$

$$\langle Z^{(j)}\rangle = e^{i\psi}\ \langle Z^{(j-1)}\rangle, \qquad (2.27a)$$

From the above equation,

$$\langle Z^{(2)}\rangle = e^{i\psi}\ \langle Z^{(1)}\rangle$$

$$\langle Z^{(3)}\rangle = e^{2i\psi}\ \langle Z^{(1)}\rangle$$

$$\vdots \qquad\qquad \vdots$$

$$\langle Z^{(N)}\rangle = e^{i(N-1)\psi}\ \langle Z^{(1)}\rangle$$

In general, the above relations can be written as,

$$\langle Z^{(j)}\rangle = e^{i(j-1)\psi}\langle Z^{(1)}\rangle \qquad 2 \le j \le N+1 \qquad (2.27b)$$

if $j=N+1$,

$$\langle Z^{(N+1)}\rangle = e^{iN\psi}\ \langle Z^{(1)}\rangle$$

But, $\langle Z^{N+1}\rangle = \langle Z^{(1)}\rangle$, therefore,

$$e^{iN\psi} = 1 = e^{i2n\pi}$$

from which,

$$\text{or } \psi = 2\pi n/N \qquad (2.28)$$

2.3.3 Values of 'n'

The ψ, defined in equation (2.28), may have N possible values. The n depends on a particular mode shape under consideration and is a measure of periodicity of the mode. If $n \neq 0$ and N/n is an integer then mode shapes repeats after N/n substructures. For example if n=2, N=8, i.e., N/n = 4 means,

$$u^{(5)} = u^{(1)}$$

$$u^{(6)} = u^{(2)}$$

$$u^{(7)} = u^{(3)}$$

$$u^{(8)} = u^{(4)}$$

As the no of independent value of n is N, ψ can have the following independent values.

For even values of N :

$-2\pi(N/2-1)/N,\ ..\ -4\pi/N,\ -2\pi/N,\ 0,\ 2\pi/N,\ 4\pi/N, ...2\pi(N/2-1)/N$

For odd values of N :

$-2\pi/(N-1)/2N,\ ..-2\pi/N,\ 0,\ 2\pi/N,\ ....2\pi(N-1)/2N$

The negative values of ψ correspond to anti clockwise rotation and positive values to clockwise rotation. All the anticlockwise eigenvectors are orthogonal pairs clockwise ones. Once the clockwise vectors {Z} are known, anticlockwis vectors can be generated simply by taking complex conjugate of {Z} and

then generating eigen vectors for complete structure using $\{\bar{Z}\}$. Eigen value equation is therefore not required to be solved for negative values of n.

ψ will therefore have the following values,

For N even

$0,\ 2\pi/N, 4\pi/N,\ \ldots\ldots\ldots\ \pi$

For N odd

$0,\ 2\pi N,\ 4\pi/N,\ \ldots\ldots\ldots\ 2\pi/(N-1)/2N.$

2.3.4 Method of Complex Constraints

The generalized eigenvalue equation for free vibrations of structures is equation (1.1) i.e.,

$$([K] - \omega^2[M]) \{u\} = \{0\}.$$

The complex eigen vector $\{Z\} = \{u\} + i \{\bar{u}\}$ is also a solution of the above equation i.e.,

$$([K] - \omega^2[M]) \{Z\} = \{0\} \qquad (2.29)$$

In this section, the above equation will be reduced to that involving stiffness and mass matrices of the first substructure only.

In general, a substructure may be connected to a number of other substructures. This number can be two in the simplest case or it may include all other substructures. Here the case in

which each substructure is connected to only two neighbouring substructures, has been considered. The common node between $(j-1)^{th}$ and j^{th} substructure is considered to be a part of j^{th} substructure. Similarly the common node between j^{th} and $(j+1)^{th}$ substructure belongs to $(j+1)^{th}$ substructure. The stiffness and mass matrices for each substructure can be partitioned into a number of submatrices. The j^{th} substructure contributes only to $K^{(j)}_{j\,j}$, $K^{(j)}_{j\,j+1}$, $K^{(j)}_{j+1\,j}$, $K^{(j)}_{j+1\,j+1}$, with similar expressions for mass matrices. Similarly $(j-1)^{th}$ substructure contributes to $K^{j-1}_{j-1\,j-1}$, $K^{(j-1)}_{j-1\,j}$, $K^{(j-1)}_{j\,j-1}$, and $K^{(j-1)}_{j\,j}$. The assembled stiffness matrix of $(j-1)^{th}$ and j^{th} substructure will look like,

$$\left[\begin{array}{c|c|c}
 & & \\
\hline
K^{(j-1)}_{j-1\,j-1} & K^{(j-1)}_{j-1\,j} & \\
\hline
K^{(j-1)}_{j\,j-1} & K^{(j-1)}_{j\,j} + K^{(j)}_{j\,j} & K^{(j)}_{j\,j+1} \\
\hline
 & K^{(j)}_{j+1\,j} & K^{(j)}_{j+1\,j+1} \\
\hline
 & & \\
\end{array}\right]$$

The diagonal submatrices will thus have contributions from two substructures and off diagonal submatrices will have

contributions from one substructure only. The stiffness matrix for the complete structure can be written as,

$$\begin{bmatrix} K_{11}^{(1)}+K_{11}^{(N)} & K_{12}^{(1)} & & -0- & & K_{1N}^{(N)} \\ K_{21}^{(1)} & & & & & \\ & & K_{j-1\,j-1}^{j-1} & K_{j-1\;j}^{j-1} & & | \\ | & K_{j\;j-1}^{j-1} & K_{jj}^{j-1}+K_{jj}^{j} & K_{j\,j+1}^{j} & & 0 \\ 0 & & K_{j+1\;j}^{j} & K_{j+1\,j+1}^{j} & & | \\ | & & & & & \\ K_{N1}^{(N)} & & -0- & & K_{NN-1}^{(N-1)} & K_{NN}^{(N-1)}+K_{NN}^{(N)} \end{bmatrix} \quad (2.30)$$

Similarly mass matrix for the complete structure can be formed.

The vector $\{Z\}$ in terms of subvectors can be written as,

$$\{Z\} = \{ \{Z^{(1)}\} \quad \{Z^{(2)}\} \ldots\ldots\ldots\ldots\ldots \{Z^{(N)}\} \}^T \quad (2.31)$$

From equations (2.27b), $\{Z\}$ in terms of first the substructure can be written as,

$$\{Z\}=\{\{Z^{(1)}\} \quad e^{i\psi}\{Z^{(1)}\} \quad e^{2i\psi}\{Z^{(1)}\}\ldots e^{i(N-1)\psi}\{Z^{(1)}\}\}^T \quad (2.32)$$

Consider equation (2.29) with [K] and similarly [M] given by equation (2.30) and $\{Z\}$ by equation (2.32). The first row of equation (2.29), can be written as,

$$\left(\left[K_{11}^{(1)}+K_{11}^{(N)} \quad K_{12}^{(1)} \quad -0- \quad K_{1N}^{(N)}\right] -\omega^2\left[M_{11}^{(1)}+M_{11}^{(N)} \quad M_{12}^{(1)} \quad -0- \quad M_{1N}^{(N)}\right]\right) \begin{Bmatrix} Z^{(1)} \\ e^{i\psi} Z^{(1)} \\ \vdots \\ e^{i(N-1)\psi} Z^{(1)} \end{Bmatrix} = 0 \quad (2.33)$$

On simplification, equation (2.33) can be reduces to,

$$\langle [(K_{11}^{(1)}+K_{11}^{(N)})+e^{i\psi} K_{12}^{(1)} +e^{i(N-1)\psi} K_{1N}^{(N)}]$$

$$-\omega^2[(M_{11}^{(1)}+M_{11}^{(N)})+e^{i\psi}M_{12}^{(1)}+e^{i(N-1)\psi} M_{1N}^{(N)}]\rangle \{Z^{(1)}\}=0$$

(2.34)

Consider $e^{i(N-1)\psi}$,

$$e^{i(N-1)\psi} = e^{iN\psi} e^{-i\psi}$$

$$= e^{-i\psi} \tag{2.35}$$

To reduce the equation (2.34) to that involving subvector $\langle Z^{(1)}\rangle$ only, it is required to find the relation between the submatrices of the first and the last substructure. Since all the substructures are identical,

$$K_{11}^{(N)} = K_{22}^{(1)}$$
$$K_{1N}^{(N)} = K_{21}^{(1)}$$
$$K_{N1}^{(N)} = K_{12}^{(1)}$$

Similar relations apply to mass matrices also. Using the above relations and equation (2.35), equation (2.34) reduces to,

$$\langle [(K_{11}^{(1)}+K_{22}^{(1)})+e^{i\psi} K_{12}^{(1)} +e^{i(N-1)\psi} K_{21}^{(1)}]$$

$$-\omega^2[(M_{11}^{(1)}+M_{22}^{(1)})+e^{i\psi}M_{12}^{(1)}+e^{i(N-1)\psi} M_{21}^{(1)}]\rangle \{Z^{(1)}\}=0$$

(2.36)

Equation (2.36) is an equation involving stiffness and mass matrix of first substructure only. This equation can be solved to give complex eigen vector $\langle z^{(1)}\rangle$. Eigen vectors for the complete structure can be generated in the following way:

Using the equation (2.27b) and (2.28) and $\{Z\}^{(1)}$ the eigenvectors for all the substructures are obtained. Thus the eigenvector for complete structure $\{Z\}$, is obtained. From equation (2.25) the real part of this vector is $\{u\}$ and imaginary part is $\{\bar{u}\}$. The two orthogonal vectors are thus obtained.

It may be seen that, both stiffness and mass matrices, appearing in equation (2.36) are hermitian, i.e., real parts are symmetric and imaginary parts are skew symmetric. It is a property of hermitian matrices that the eigen values are real but the eigen vectors are complex. The solution of equation (2.36) will therefore give real values of ω^2 and complex eigen vector $\{Z^{(1)}\}$.

[M] ORTHONORMALISATION

The eigen vectors can be made [M] orthonormal by finding the quantity $\alpha=\{u\}^T[M]\{u\}$ and then dividing the the eigenvectors by square root of this number. The α can be obtained using the mass matrix of the first substructure only. For finding α, the relationship between the submatrices of substructures and that between the subvectors can be used. For the cases in which each substructure is connected to only two substructure, the expression for α is derived as follows:

Let $\{Z^{(1)}\} = X + i\ Y$

From equations (2.27b) and (2.28), the eigen vector for complete structure $\{u\}$ is given by,

$$\{u\}=\{X \quad X\cos\psi-Y\sin\psi \quad X\cos2\psi-Y\sin2\psi \ \ldots X\cos(N-1)\psi-Y\sin(N-1)\psi\}^T$$

Since all substructures are identical i.e.,

$$M_{11}^{(1)} = M_{22}^{(2)} = M_{33}^{(3)} \ \ldots\ldots = M_{NN}^{(N)},$$

$$M_{12}^{(1)} = M_{23}^{(2)} = M_{34}^{(3)} \;.....= M_{N1}^{(1)} \quad \text{and}$$
$$M_{21}^{(1)} = M_{32}^{(2)} = M_{43}^{(3)} \;.....= M_{1N}^{(N)}$$

the expression for $\alpha = \{u\}^T[M]\{u\}$, after simplification comes out to be

$$\alpha=\sum_{j=1}^{N} \{X\cos(j-1)\psi - Y\sin(j-1)\}^T(M_{11}^{(1)}+M_{22}^{(1)})\{X\cos(j-1)\psi - Y\sin(j-1)\psi\}$$

$$+\sum_{k=0}^{N-1}\sum_{j=2}^{N} \{X\cos k\psi - Y\sin k\psi\}^T M_{12}^{(1)} \{X\cos(j-1+k)\psi - Y\sin(j-1+k)\psi\}$$

$$+\sum_{k=0}^{N-1}\sum_{j=2}^{N} \{X\cos k\psi - Y\sin k\psi\} M_{21}^{(1)} \{X\cos(k+1-j)\psi - Y\sin(k+1-j)\psi\} \tag{2.37}$$

Similar expression is obtained for $\{\bar{u}\}$ also.

2.3.5 Computer time and storage

It is interesting to compare the computer time and storage required in a solution of the eigenvalue problem of a structure by the two methods. To find the solution by standard finite element method the equation (1.1) is solved whereas while using the method of complex constraints the equation (2.36) is solved. The major part of the solution is finding the eigenvalues and eigen vectors, once the stiffness and mass matrices have been set up. Most of the subroutines take the CPU time proportional to the cube of the number of degrees of freedom. To solve equation (1.1) the computational time will be proportional to N^3J^3. While using the method of complex constraints only J degrees of freedom will be used, but multiplications involved are complex

and each complex multiplication is equivalent to four real multiplications. Therefore, the time required to solve equation (2.36), for one value of n, will be proportional to $4J^3$. To analyse a problem completely the equation (2.36) is required to be solved for (N/2 + 1) times for N even (which is the worse case) so the computational time will be proportional to (N/2 + 1) $4J^3$ or $(2N + 4)J^3$. The ratio of CPU time required to solve the equation (1.1) and that required for solving equation (2.36) will be,

$N^3J^3/(2N + 4)J^3 = N^3/(2N+4)$.

For N=4 this ratio will be 5.33, i.e. the computation time required while using standard finite element is 5.33 times more than that taken by the method of complex constraints.

CHAPTER III

RESULTS AND DISCUSSIONS

As discussed in the earlier chapters, two problems have been solved to test the efficacy of the method of complex constraints. The first problem analyzed is that of free vibrations of a simplified version of a banded blade structure, whereas the second problem is that of two span beam with three different end conditions. Results obtained by the method of complex constraints(method 1) and those obtained by the finite element assembly for complete structure (method 2) have been compared.

3.1 Banded Blades

Idealizations made for the banded blade problem have been discussed in the first chapter under the heading "Scope of the present work". The simplified structure consists of equally spaced four cantilever beams which are connected as a chain (fig 3.1a). The structure is divided into four substructures. The number of substructures could have been chosen as two or three also but two substructures are too less to understand the potential and behaviour of rotationally periodic structures. Three substructure have not been chosen because 3 being an odd number, antisymmetric modes can not be obtained. Four substructures give all the three classes of modes, discussed

in chapter II. Though it would have been better to include more than four substructure but as it happened to be the first problem it was decided to restrict to four substructures. The substructure selected is shown in fig.(3.1b). The structure is cyclic periodic i.e. the deflection, the slope and the curvature are same at the first point of the first substructure and the last point of the fourth substructure.

3.1.1 IMPLEMENTATION DETAILS

While solving by the substructuring method (the method of complex constraints), the finite element assembly for the first substructure only is required, whereas while solving by the second method (method 2) the finite element assembly for complete structure i.e. all four substructures is required. For the plane vibrations of the structure and hence the substructure, each node will have three degrees of freedom namely, u,v and θ. The elemental stiffness and mass matrices are of the following type,

$$[k]^{(e)} = \left[\begin{array}{c|c} [k]_a & 0 \\ \hline 0 & [k]_b \end{array}\right]$$

$$[m]^{(e)} = \left[\begin{array}{c|c} [m]_a & 0 \\ \hline 0 & [m]_b \end{array}\right]$$

where $[k]_a$ and $[m]_a$, which are element stiffness and mass matrices respectively for axial vibrations, are given by equation (2.7) and $[k]_b$ and $[m]_b$, which are element stiffness and mass

matrices for bending vibrations are given by equations (2.13) and (2.14). The local displacement vector is of the form,

$$\{u\}^{(e)} = \{ u_1 \quad u_2 \quad v_1 \quad \theta_1 \quad v_2 \quad \theta_2 \}^T$$

and the global displacement vector is of the following type,

$$\{u\} = \{u_1 \quad v_1 \quad \theta_1 \quad u_2 \quad v_2 \quad \theta_2 \quad \ldots\ldots\ldots\ldots u_n \quad v_n \quad \theta_n \}^T$$

The element stiffness and mass matrices have been transformed accordingly before assembly. The transformed local displacement vector is defined by,

$$\{u^{(et)}\} = [T] \{u\}^{(e)},$$

where [T] is given by,

for horizontal elements,

$$T= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

for vertical elements,

$$T= \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed element stiffness and mass matrices are given by,

$[k] = [T]^T [k]^{(e)} [T]^T$, and

$[m] = [T]^T [m]^{(e)} [T]^T$.

The material properties and geometry details are as follows:

$E = 200 \times 10^9 \quad N/m^2$

$\rho = 7800 \quad Kg/m^3$

$\frac{I}{A} = 1.0 \times 10^{-4} \quad m^2, \quad l_h = 1\ m \ \&\ l_v = 0.3\ m$

3.1.2 Comparison of results obtained by the two methods

The number of substructures in this case is 4, which is an even number. As discussed in chapter II, for even values of N, the independent values of n vary from 0 to N/2. So for N = 4, the independent values of n will be 0, 1 and 2. Equation (2.36) is therefore solved for these three values of n. Table 1 gives first few natural frequencies of each case.

According to equation (2.28), $\psi = 2\pi n/N$.

For $n=0$ $\psi=0$ and

for $n=2$ $\psi=\pi$.

The cases of n=0 and n=2 therefore represent symmetric (class (a)) and antisymmetric (class (b)) modes respectively. The n=1 case contains class (c) modes i.e. orthogonal pairs. Natural frequencies corresponding to n=1 case are therefore repeated eigen values. Table 2 gives the comparison of natural frequencies obtained by the two methods.

From Tables 1 and 2, the following points are observed.

i) The frequencies obtained corresponding to the cases of n=0, 1 and 2 include all the frequencies obtained by method 2

ii) Frequencies obtained by method 2, which match with those of n=1 case are repeated eigenvalues.

The mode shapes, obtained by sustructuring method, are [M] orthogonal but not [M] orthonormal. [M] orthonormality property is important from modal analysis point of view. The eigen vectors have been made [M] orthonormal by finding the quantity $\alpha=\{u\}^T[M]\{u\}$, using the expression (2.37) and then dividing the the eigenvectors by squre root of this number.

Figures(3.2-3.10) show first three modes of each case of n. Figures (3.11-3.23) show the first thirteen modes obtained by method 2. Mode shapes corresponding to the cases of n=0 and n=2 matches with the corresponding modes obtained by method 2. The two orthogonal modes of the case of n=1 not match with corresponding modes obtained by method 2. But this is the case of repeated eigen values and it is well known that in such cases eigen vectors can not be represented uniquely. The eigen vectors in such cases lie in a 2-D subspace of n-dimensional space and that it is always possible to select a set of two orthogonal vectors which span the 2-D subspace. It follows that any vector lying in the 2-D subspace can be represented as a linear combination of the two orthogonal vectors. The substructuring method for n=1 case directly gives two [M] orthogonal vectors $\{u\}$ and $\{\bar{u}\}$. The first natural frequency of n=1 case matches with the third and the fourth natural frequency of method 2. It is observed that the following of the two eigen vectors corresponding to the first eigen value of the case n=1 match with

the 3rd and 4th eigen vectors obtained by method 2:

$0.0946010 \langle u \rangle - 0.2757681 \langle \bar{u} \rangle$ = 3rd eigen vector and

$0.2748304 \langle u \rangle + 0.9492391 \langle \bar{u} \rangle$ = 4th eigen vector.

Similarly, linear combinations can be made for other cases also.

3.2. Finite continuous beam

The second problem attempted is 2-span beam with the following three set of end conditions.

(a) Both extreme ends hinged (Type HHH).

(b) Both extreme ends fixed (Type FHF).

(c) One end fixed, other hinged (Type FHH).

3.2.1 Construction of equivalent cyclic periodic structure

Case 1 Both ends hinged (type-HHH)

Figures (3.24a),(3.24b) and (3.24c) the beam HHH and it's first two possible modes. Consider the first mode. It can be seen that the deflection, the slope, and the curvature are same at the two extreme ends. The conditions for cyclic periodicity is thus satisfied. It is observed that all the higher order odd modes have similar end conditions and hence they also satisfy the conditions for cyclic periodicity. Now, consider the second mode. It can be seen that the slope at the two extreme ends is not same. So the condition for cyclic periodicity is not satisfied. Similar is the situation with all higher order even modes. The method of complex constraint can not therefore be applied to this beam (HHH). However, it is possible to construct

an equivalent cyclic periodic structure. The method of forming such a structure is discussed below.

Consider again the second mode. In order construct an equivalent cyclic periodic structure, it is required to extend the original two span beam by its anti-symmetric image (fig 3.27), so that the continuity of slope and bending moment (curvature) is maintained. It will result in a four span beam resting on four hinged supports and which is a cyclic periodic structure. In this cyclic periodic structure thus obtained, the the deflection, the slope and the curvature are the same at the two extreme ends. Thus , to get the natural frequencies and mode shapes of the two span beam resting on three hinge supports, the four span beam should be analyzed while using the method of complex constraints.

Case 2 Both ends fixed (type FHF)

The first two modes of this case are shown in fig (3.25b)and (3.25c). For the first mode, the slope and the deflection are same, both of these quantities being zero, at the two extreme ends. But, the curvature at the two ends are not same. Therefore the condition for cyclic periodicity is not satisfied. Similar is the case for all other odd modes. The second mode however satisfies the condition of cyclic periodicity. It is noticed that all the even modes satisfy the condition of cyclic periodicity. To analyse the two-span beam with this kind of end conditions, using the method of complex constraints, here also the equivalent cyclic periodic structure can be constructed. To

obtain the equivalent cyclic periodic structure it is required to extend the original beam by its anti symmetric image (fig 3.28). This will result in a four span beam. It should be noted here that in order to be cyclic periodic, the equivalent structure has to have hinged support at all the four places. However, because of symmetry, certain mode shapes will have zero slopes at two or all the four supports. Such modes will belong to the case under consideration. The remaining modes will belong to the case 1 i.e., the HHH beam.

Case 3 One end fixed, other hinged (type-FHH)

In this case, one end is fixed and other is hinged. For every mode the slope at the fixed end will be zero and that at the extreme hinged end will be a non zero quantity. Therefore the condition for cyclic periodicity will never be satisfied for any mode. Figures (3.26b) show the first possible mode of this case. Here also an equivalent cyclic periodic structure can be constructed. For this purpose the original beam is first extended by its anti symmetric image. This will result in a four span beam with end conditions similar to odd modes of both ends fixed (FHH) beam. If this intermediate structure is again extended by its anti symmetric image, the structure obtained will be a cyclic periodic structure consisting of eight spans (fig 3.29). Again, it should be noted that for the resulting structure to be rotationally periodic, all the supports should be hinged. As stated earlier because of symmetry the slope at the two, four or all the supports will have zero values. Some of

these modes will belong to this type of beam.

It is thus observed that for the first two cases the equivalent structure consists of four spans and for the third case it is an eight span beam. But it is not required to analyze the four span and the eight span beams separately. If the eight span beam is analyzed using the method of complex constraints, it will give natural frequencies and mode shapes of all the three set of boundary conditions of the two span beam.

To take care of all the three types of end conditions of the two span beam, the continuous beam with eight spans, resting on hinge supports only, has been analyzed.

3.2.2 Implementation details

In this case only transverse vibrations have been considered. The elemental stiffness and mass matrices are given by equations (2.13) & (2.14). The local displacement vector is of the form,

$$\{v\}^{(e)} = \{\ v_1 \quad \theta_1 \quad v_2 \quad \theta_2\ \}^T$$

and the global displacement vector is of the following type,

$$\{V\} = \{\ v_1 \quad \theta_1 \quad v_2 \quad \theta_2 \quad \ldots\ldots\ldots\ldots v_n \quad \theta_n \}^T .$$

Therefore no transformation is required.

The material properties and geometry details are as follows,

$E = 200 \times 10^9$ N/m^2

$\rho = 7800$ Kg/m^3

$\frac{I}{A} = 1.0 \times 10^{-4}$ m^2

Length of each span = 1 m.

Natural frequencies and mode shapes for eight-span cyclic periodic continuous beam have been obtained by substructuring method(method of complex constraints). The substructure is shown in fig.(3.30). To compare the results, natural frequencies and mode shapes for the two span beam have been obtained by complete FEM assembly for the two span beam.

3.2.3 Comparison of results obtained by the two methods

In this case, the number of substructures are eight i.e. N=8, so the independent values of n will be,

n=0, 1, 2, 3 & 4.

Table 3 gives first few eigenvalues corresponding to various cases of n. Table 4 enlists first few eigenvalues of the two span beams for the three different types of end conditions. Table 5 gives the comparison of natural frequencies of the two span beam with both extreme ends hinged (type HHH) case and the corresponding frequencies of equivalent cyclic periodic structure, i.e. the eight span beam. Similarly Tables 6 and 7 give the comparison of natural frequencies of the two span beam for the other two sets of end conditions (type FHF & FHH), and the corresponding frequencies of equivalent cyclic periodic structure.

The following points are be observed from Tables 3-7,

1. As anticipated, the five cases (n=0, 1, 2, 3, 4) of eight span beam include natural frequencies of all the three types of the two span beam.

2. The cases of n=0 and n=4 include the natural frequencies of the first two types of the two span beam only, i.e. HHH and FHF. This is because of the symmetric nature of the symmetric nature of end conditions, for these two types of the two span beam. In particular, these are odd frequencies of HHH type and even frequencies of FHF type for which the two-span beam itself is cyclic periodic.

3. The natural frequencies of the case of n=2 also happen to match with the frequencies of HHH and FHF cases only. The case include those frequencies which are common to the type FHF and HHH. In particular, these are even frequencies of HHH type and odd frequencies of FHF type for which the two-span beam needs to be extended to make it cyclic periodic.

4. The cases of n=1 and n=3 include all the natural frequencies of the third case of the two-span beam, i.e., FHH type.

Mode shapes obtained in this case also are [M] orthogonal but not [M] orthonormal. They have been made [M] orthonormal using equation (2.37).

Mode shapes obtained for eight span beams have been compared with corresponding modes of the two span beam by extending the modes of the two span beam over eight spans. Figures (3.31-3.40) show the first two modes corresponding to the five cases of n, including orthogonal modes. Figures (3.41-3.43) show first four modes of each types of the two - span beam.

Mode shapes of the the cases of n=0 and n=4

From equation (2.28), $\psi=2\pi n/N$, which means,

for n=0, $\psi=0$ and

for n=4, $\psi=\pi$

Therefore the cases of n=0 and n=4 represent symmetric (class (a)) and anti symmetric(class (b)) modes respectively. These modes match with corresponding modes of HHH and FHF types of the two - span beam. As stated earlier the modes of the cases of n=0 and n=4 match with those modes of HHH and FHF types, for which the two span beam itself is cyclic periodic. Therefore, to compare these modes it is not required to extend these mode shapes of the two span beam, over eight spans.

Mode shapes for n=2

Mode shapes of the case of n=2 correspond to class (c) modes i.e., they are associated with repeating eigen values. As stated earlier, these modes match with those modes of symmetric end conditions for which the two span beam had to be extended in antisymmetric fashion to make it cyclic periodic, i.e. even modes of HHH type and odd modes of FHF type. The reason for this is as follows.

For n=2, $\psi=\pi/2$. Therefore, the phase difference of various substructures with the first substructure will be as follows,

Substructure	Phase angle ψ
1	0
2	$\pi/2$
3	$3\pi/2$
4	2π
5	$5\pi/2$
6	3π
7	$7\pi/2$

It can be seen from the above table that the third substructure is vibrating in antiphase with the seventh and the first substructures, while the seventh substructure is in antiphase with the first and the fifth. The fourth substructure has similr relations with the second and the sixth substructures. This type of situation can arise only in beams having symmetric end conditions.

The mode shapes of the two span beams have been extended over eight spans (fig 3.44a&b) to compare them with the corresponding modes of the eight-span beam. It is seen that the two orthogonal modes obtained by substructuring method do not match with corresponding extended modes of the two span beams. It is observed that the linear combination of the two orthogonal vectors, obtained by the method of complex constraints, match with corresponding extended vector of the two span beam. The following linear combination of the two orthogonal vectors corresponds to the first natural frequency of the case of n=2, matches with 2nd vector of HHH case and the first vector of FHF

case.

0.905879 <u> +0.2371595 <ū> = 2nd eigenvector of FHF beam

Mode shapes of the cases of n=1 and n=3

The cases of n=1 and n=3 include mode shapes of the third type of the two span beam i.e. FHH type. These modes also belong to the class c eigen vectors i.e. the case of multiple eigen values. Here also to compare the results, mode shapes of two span beam (type FHH) have been extended over eight spans (figures 3.45 and 3.46). In this case also, the modes of the two span beam and eight span beam do not match. The linear combinations have been made corresponding to first eigen values of the cases of n=1 and n=3, which match with the corresponding extended modes of the two span beams. These linear combinations are as follows:

for n=1

0.9302325 <u> + 0.178898 <ū> = 3rd eigen vector of FHH beam

for n=3

0.53095 <u> +0.47286 <ū> = 1st eigen vector of FHH beam

3.2.4 Identification procedure

It is clear from the discussion in the previous section that the equivalent eight-span beam gives the modes of all the three types of the two-span beam. Now there is a need to develop a procedure to identify that which modes of various cases of n correspond to which modes of the three types of the two-span beam, without solving the two-span beam by the standard finite

element method. The procedure is as follows,

Modes of the cases of n=0 and n=4

Careful observation of mode shapes of these two cases of n reveals that,

1. All the even modes of n=0 and odd modes of n=4 have non zero slope at each support. This situation is possible only with the first type of the two-span beam, i.e., type HHH. Therefore they correspond to HHH type beam.

2. All the odd modes of n=0 and even modes of n=4 have zero slope at each support. This situation is possible only with FHF type beam. There fore these modes correspond to the modes of FHF type beam.

Modes of the cases of n=1 and n=3

From equation 2.28, $\psi=2\pi n/N$, which means,

for n=1, $\psi=\pi/4$ and

for n=3, $\psi=3\pi/4$.

It is easy to see that any substructure is neither in phase or anti-phase with either of its neighbour. This situation will occur only in the third type of beam i.e., FHH type beam. Therefore all the modes of n=1 and n=3 correspond to the modes of FHH type beam.

Modes of the case of n=2

For n=2, $\psi=\pi/2$, which means,

$\{u\}^{(1)} = -\{u\}^{(3)} = \{u\}^{(5)} = -\{u\}^{(7)}$, and

$\{u\}^{(2)} = -\{u\}^{(4)} = \{u\}^{(6)} = -\{u\}^{(8)}$.

This type of situation is possible only with the two span beam having symmetric end conditions. So the case of $n=2$ should include the modes of FHF and HHH type of beam only. Also, $n=2$ is the case of multiple eigen values, it should therefore include those modes for which frequencies of FHF and HHH are same.

In view of the above discussion, the procedure for identifying the modes of HHH, FHF and FHH types of beam is as follows.

i) All the odd modes of $n=4$, even modes of $n=0$ and modes of $n=2$ correspond to the modes of HHH type beam. by arranging the natural frequencies of these modes in increasing order, all the modes of HHH type beam can be identified.

ii) All the even modes of $n=4$, odd modes of $n=0$ and modes of $n=2$ correspond to the modes of FHF type beam. Again by arranging the frequencies corresponding to these modes, all the the modes of FHF type beam can be identified.

iii) The modes of $n=1$ and $n=3$ include modes of FHH type beam only. So all the modes of FHH beam can be obtained by arranging the frequencies of $n=1$ and $n=3$ cases in increasing order.

$V_1 = 0$ $\qquad$ $S_n = 0$

$V_1' = 0$ $\qquad$ $M_n = 0$

fig 2.1a : Boundary conditions for a beam with one end fixed, other free

$V_1 = 0$ $\qquad$ $V_n = 0$

$M_1 = 0$ $\qquad$ $M_n = 0$

fig 2.1b : Boundary conditions for a beam with both ends hinged

$V_1 = 0$ $\qquad$ $V_n = 0$

$V_1' = 0$ $\qquad$ $M_n = 0$

fig 2.1c : Boundary conditions for a beam with one end fixed, other hinged

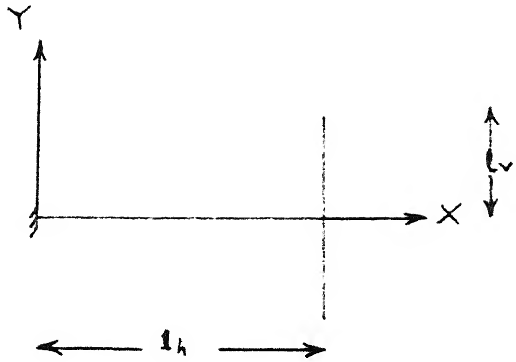


fig 3.1b The substructure selected in proble

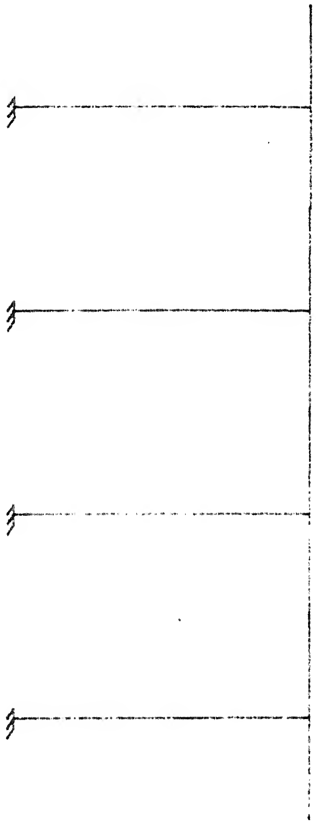


Fig 3.1a The Banded blade structure

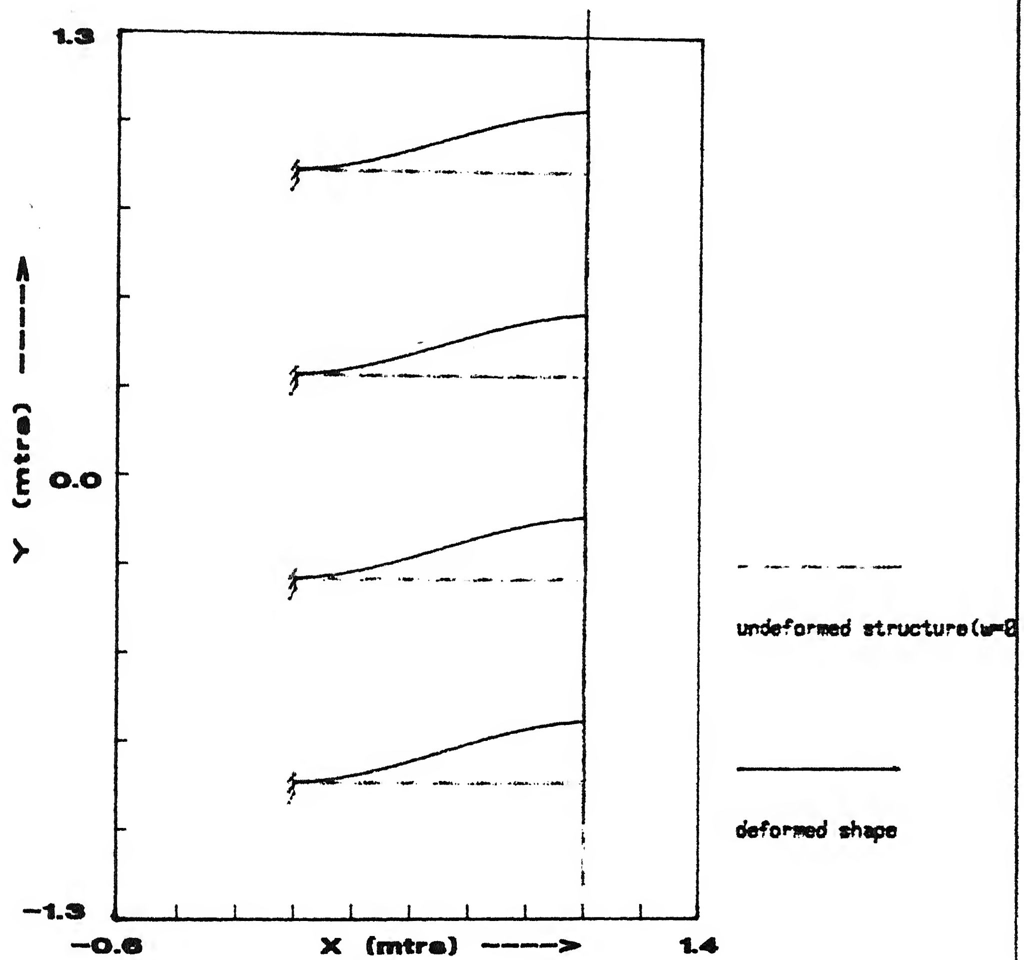


Fig.(3.2) :Mode Shape for Banded blade

1st Mode (n=0)

(w=0.01682747 E+04) Rad/sec

Solution Method :Substructuring

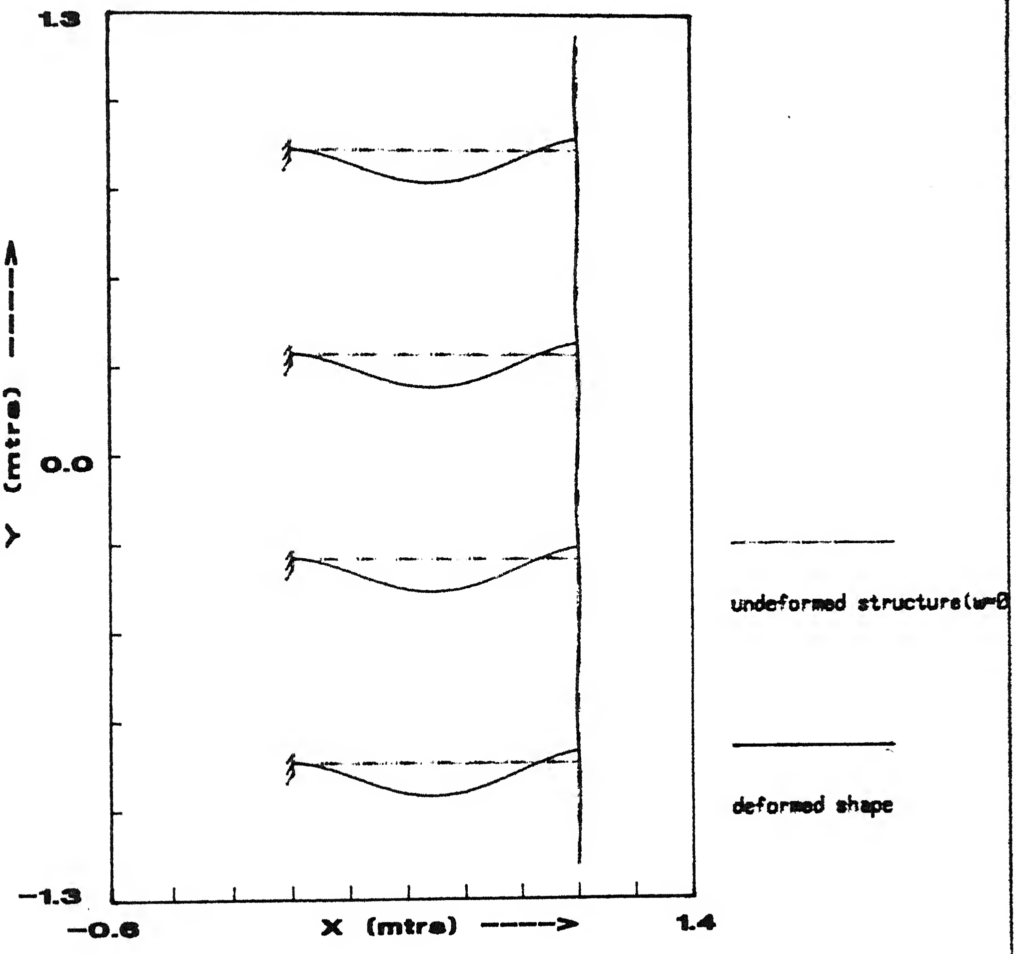


Fig.(3.3) :Mode Shape for Banded blade

2nd Mode (n=0)

(w=0.11580694 E+04) Rad/sec

Solution Method :Substructuring

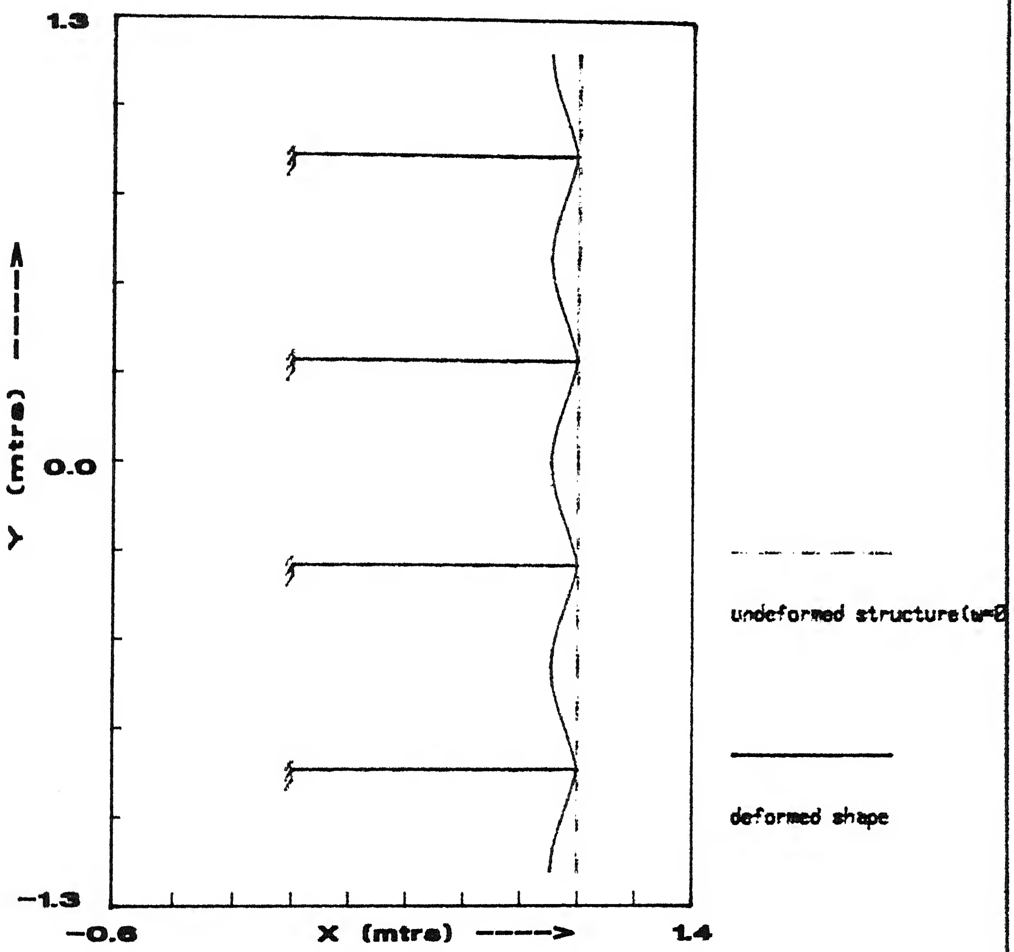


Fig.(3.4) :Mode Shape for Banded blade

3rd Mode (n=0)

(w=0.28812402 E+04) Rad/sec

Solution Method :Substructuring

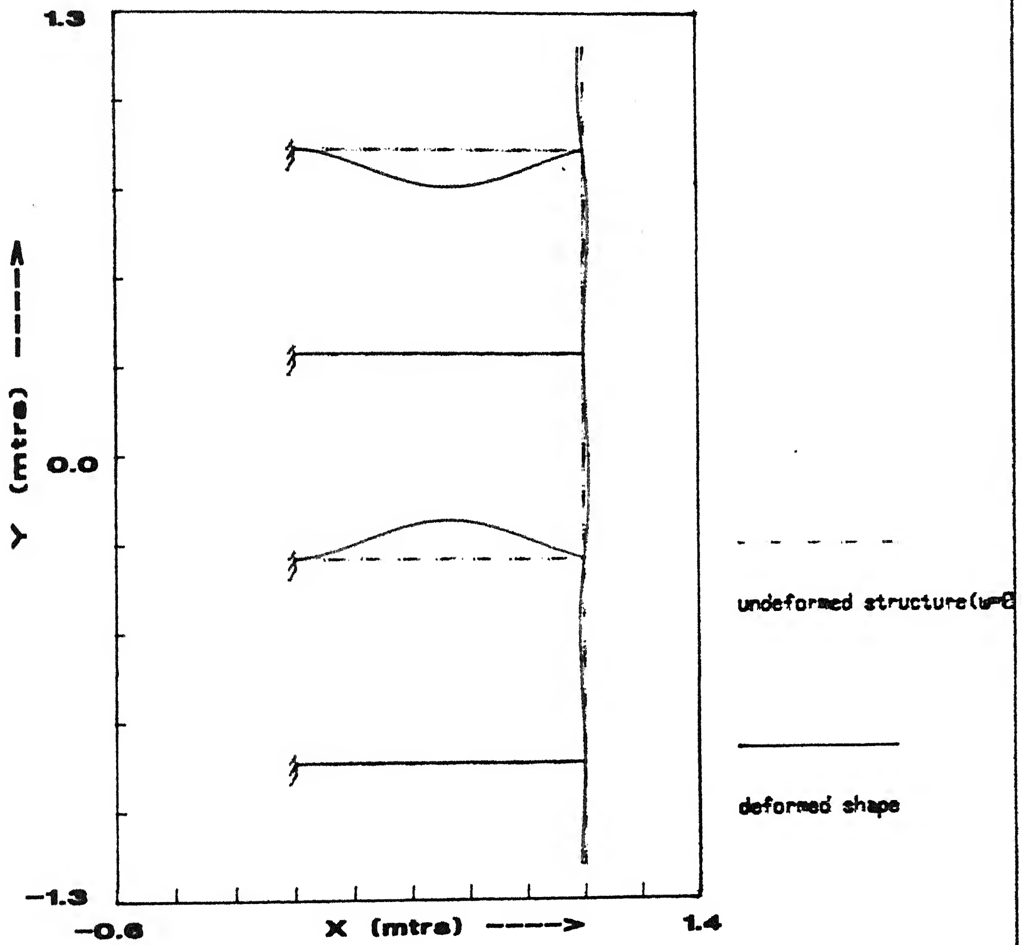


Fig.(3.5a) :Mode Shape for Banded blade

1st Mode (n=1) vector {u}

(w=0.10035578 E+04) Rad/sec

Solution Method :Substructuring

113076

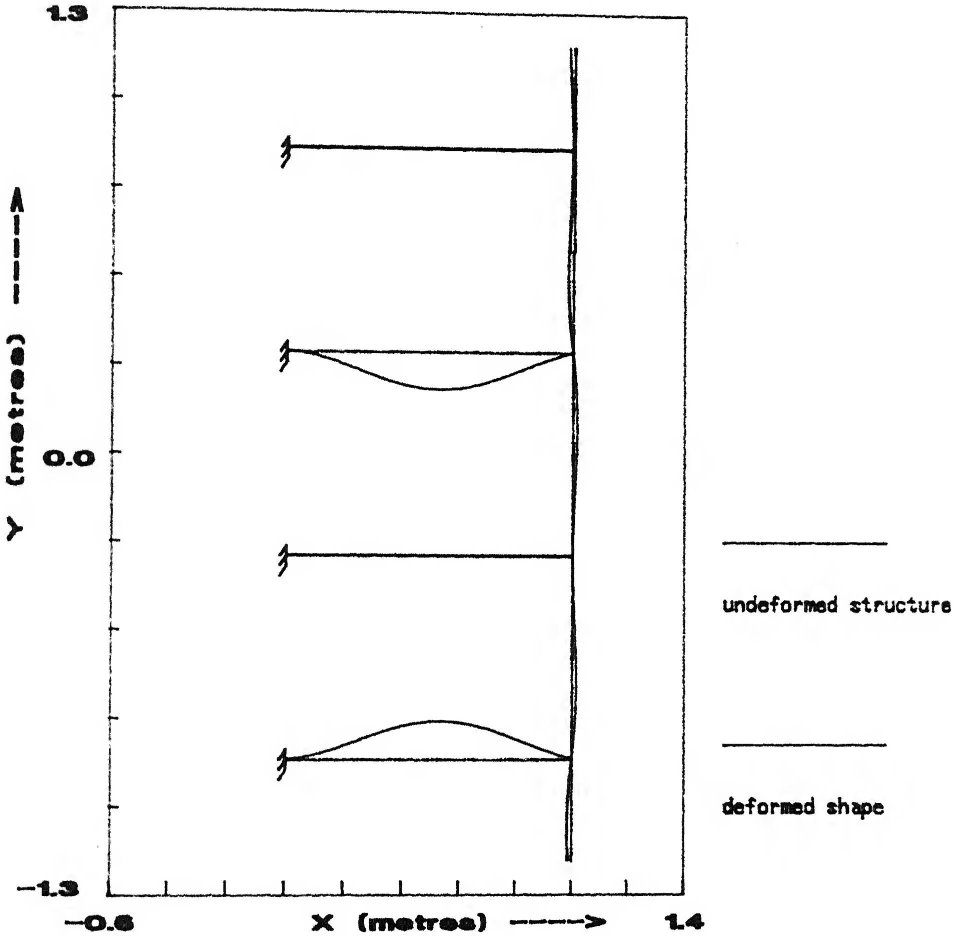


Fig.(3.5b) :Mode Shape for Banded blade

1st Mode (n=1) vector $\{\bar{u}\}$

(w=0.10035578 E+04) Rad/sec

Solution Method :Substructuring

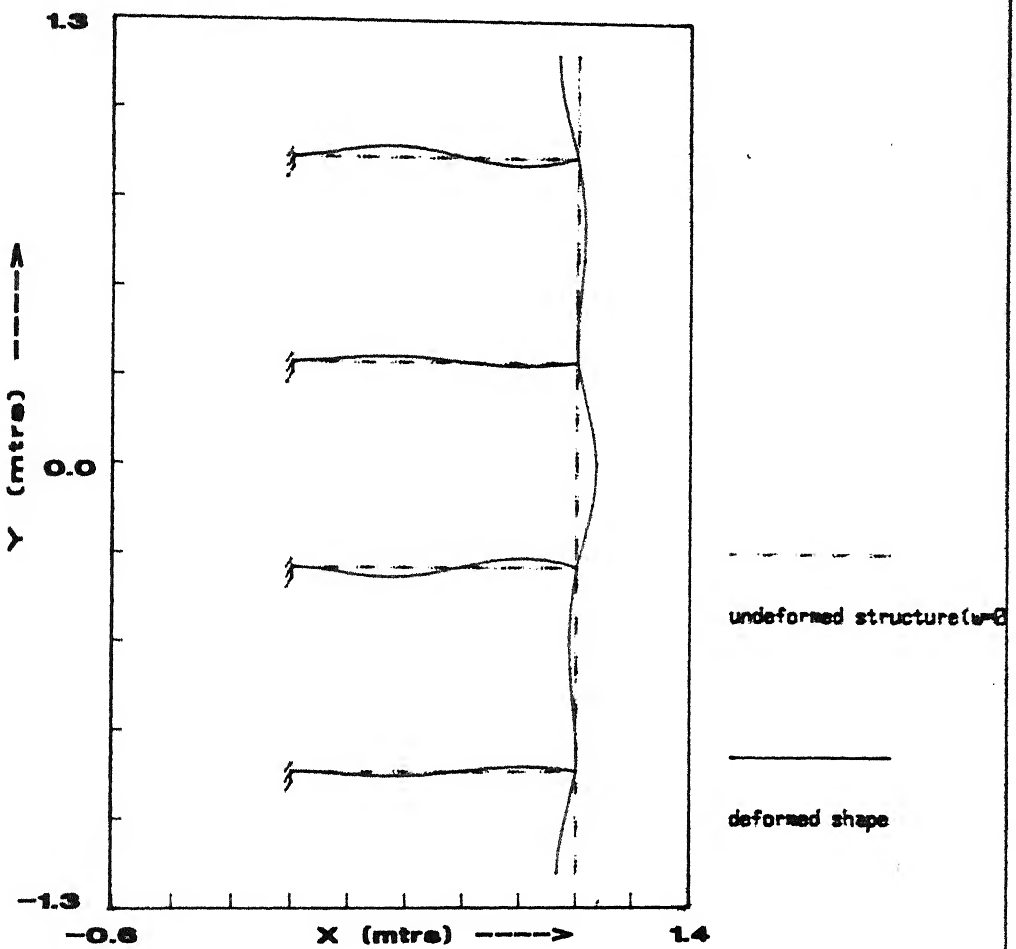


Fig.(3.6a) :Mode Shape for Banded blade

2nd Mode (n=1) vector {u}

(w=0.22468542 E+04) Rad/sec

Solution Method :Substructuring

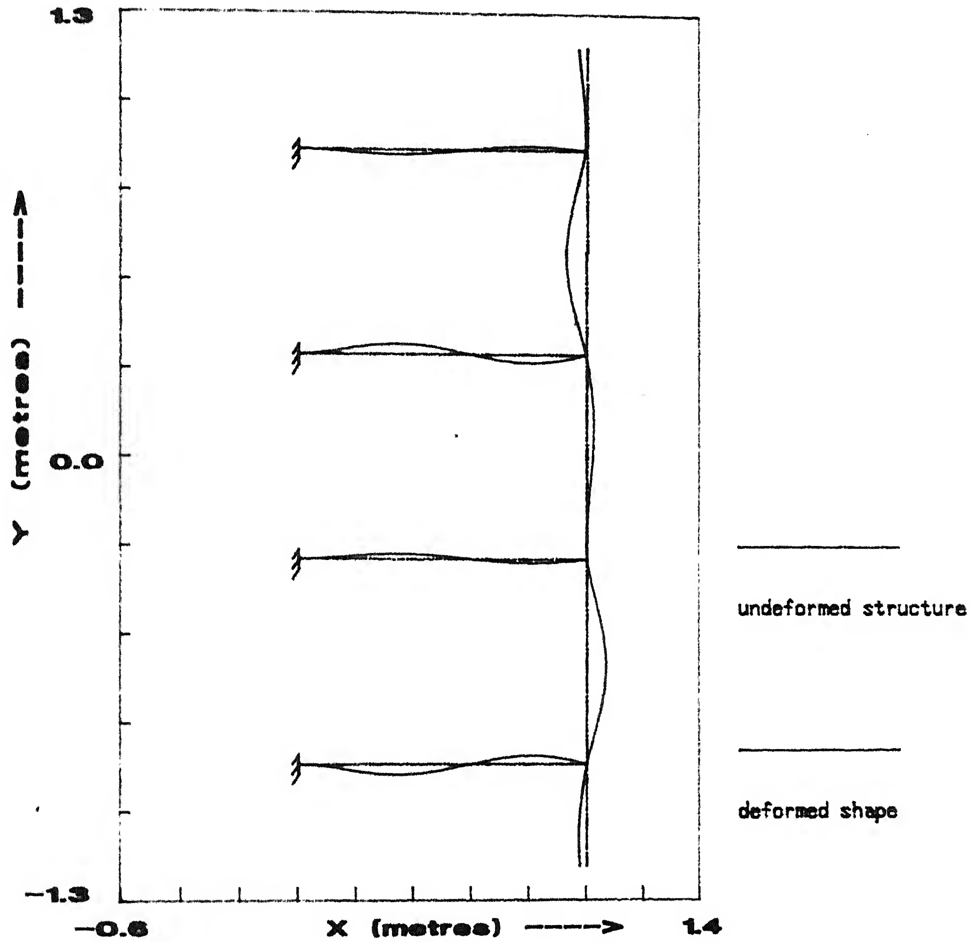


Fig.(3.6b) :Mode Shape for Banded blade

2nd Mode (n=1) vector $\{\bar{u}\}$

(w=0.22468542 E+04) Rad/sec

Solution Method :Substructuring

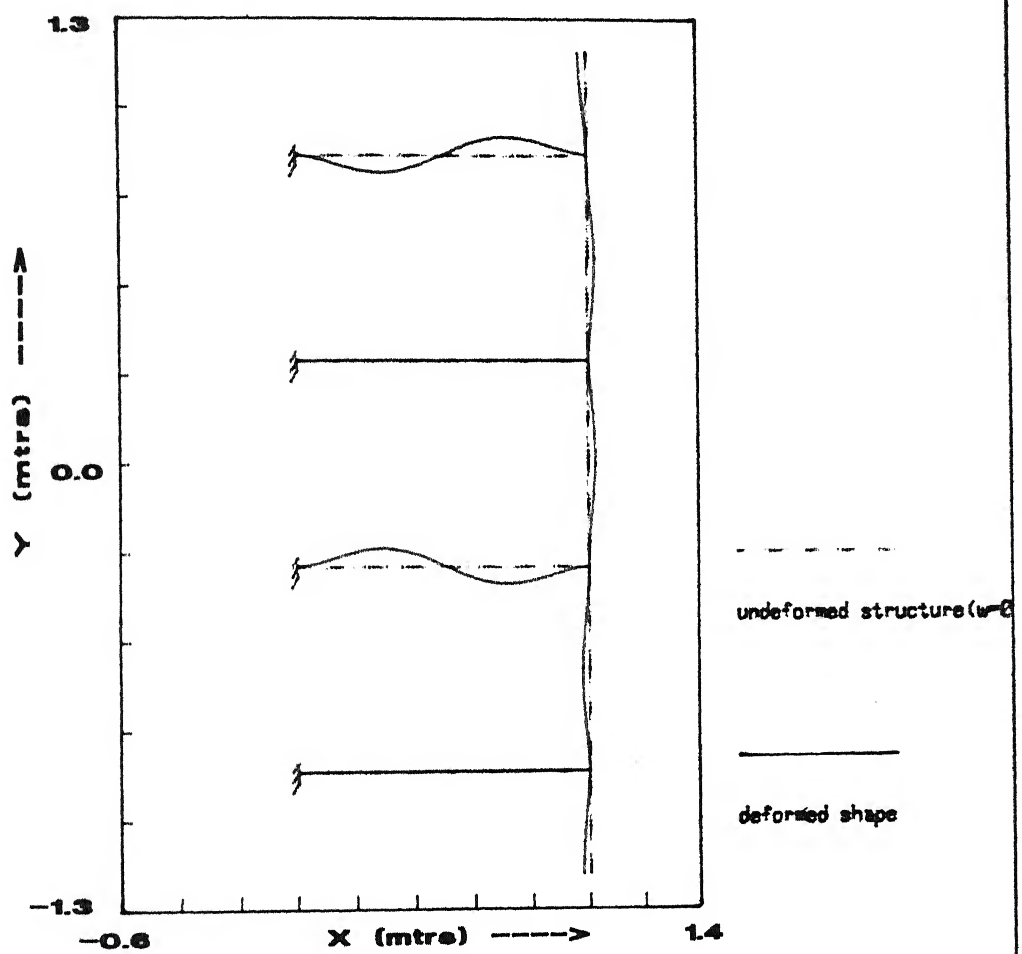


Fig.(3.7a) :Mode Shape for Banded blade

3rd Mode (n=1) vector {u}

(w=0.30856572 E+04) Rad/sec

Solution Method :Substructuring

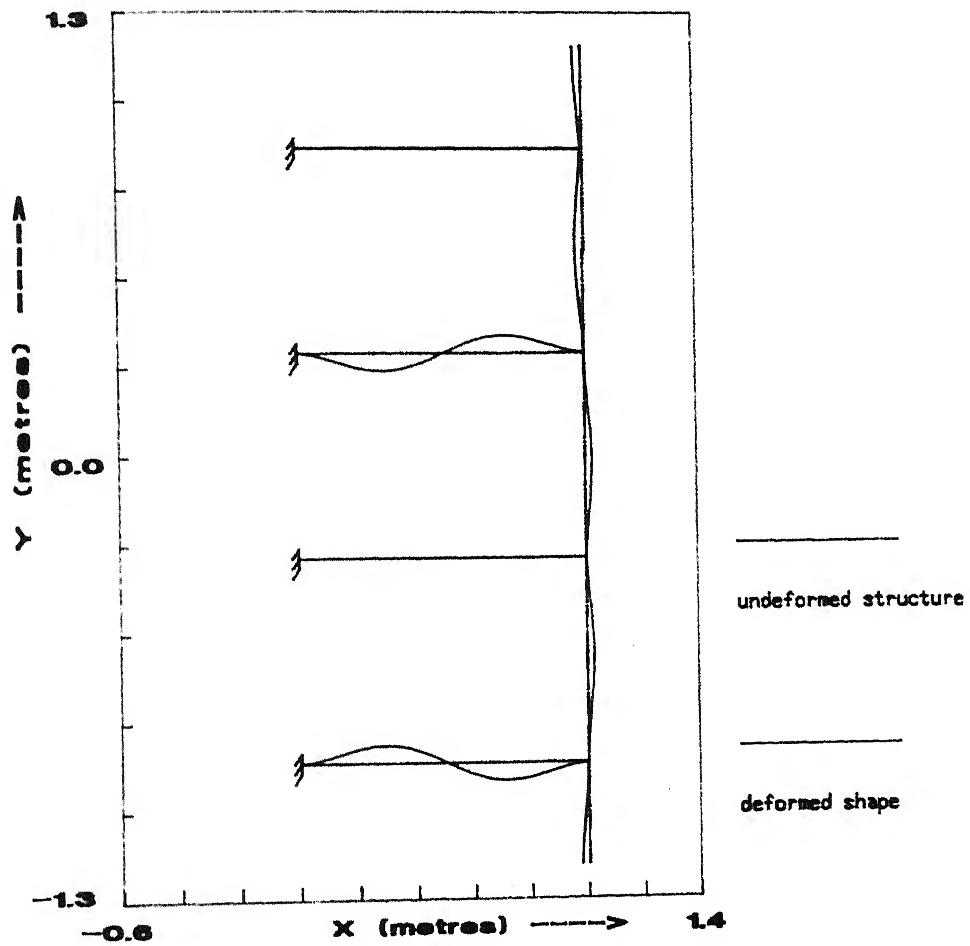


Fig.(3.7b) :Mode Shape for Banded blade

3rd Mode (n=1) vector $\{\bar{u}\}$

(w=0.30856572 E+04) Rad/sec

Solution Method :Substructuring

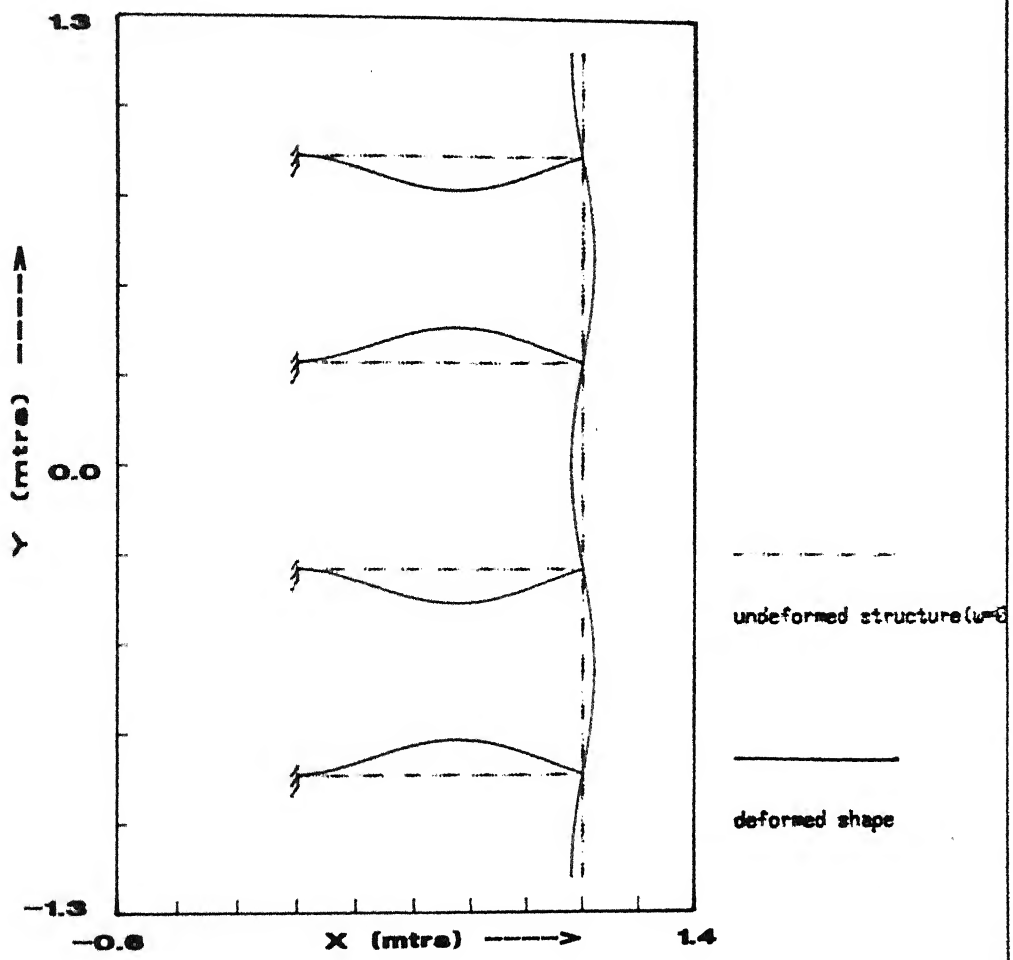


Fig.(3.8) :Mode Shape for Banded blade

1st Mode (n=2)

(w=0.09159237 E+04) Rad/sec

Solution Method :Substructuring

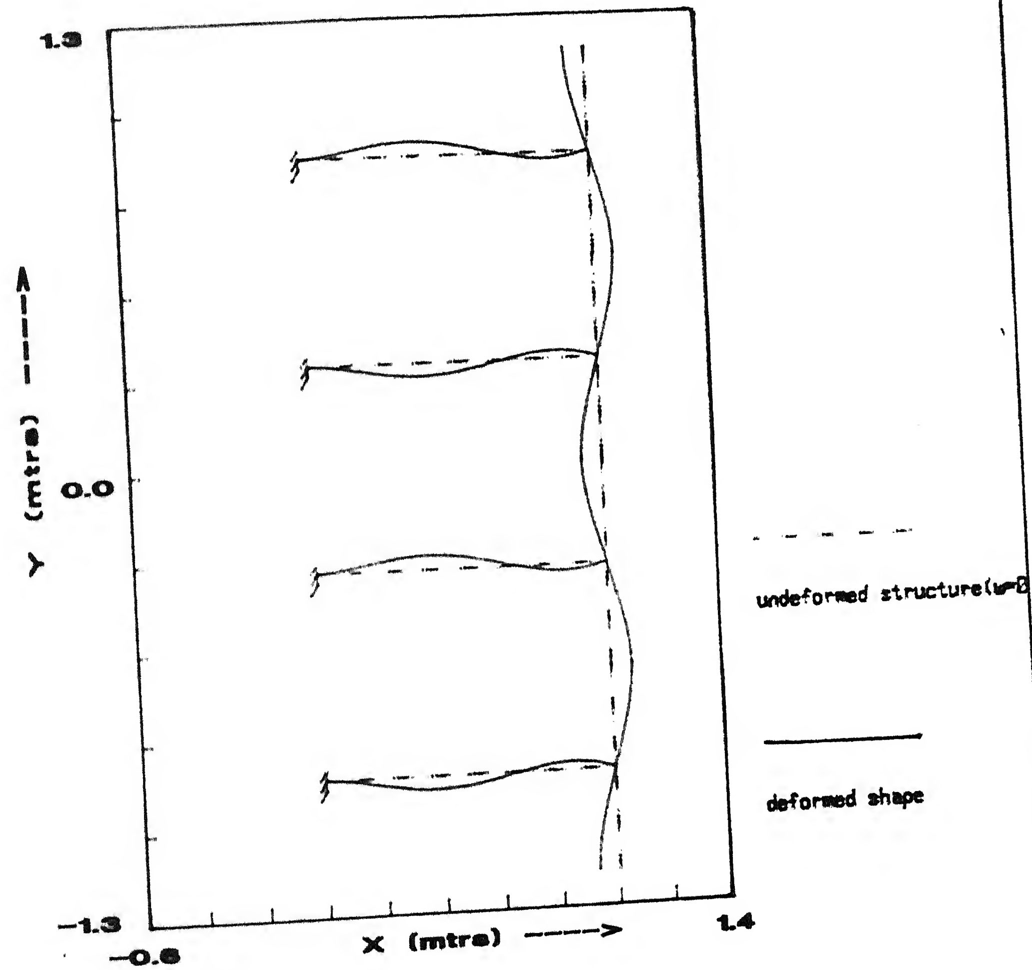


Fig.(3.9) :Mode Shape for Banded blade

2nd Mode (n=2)

(w=0.18418413 E+04) Rad/sec

Solution Method :Substructuring

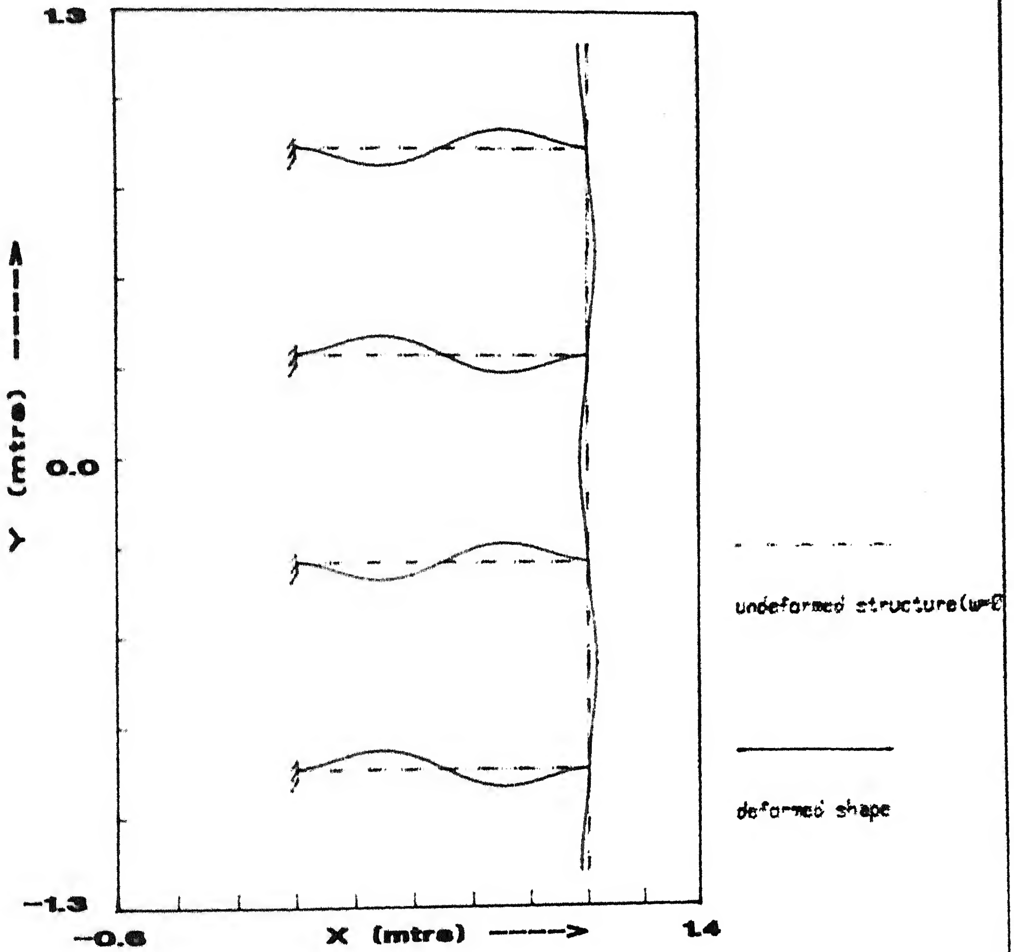
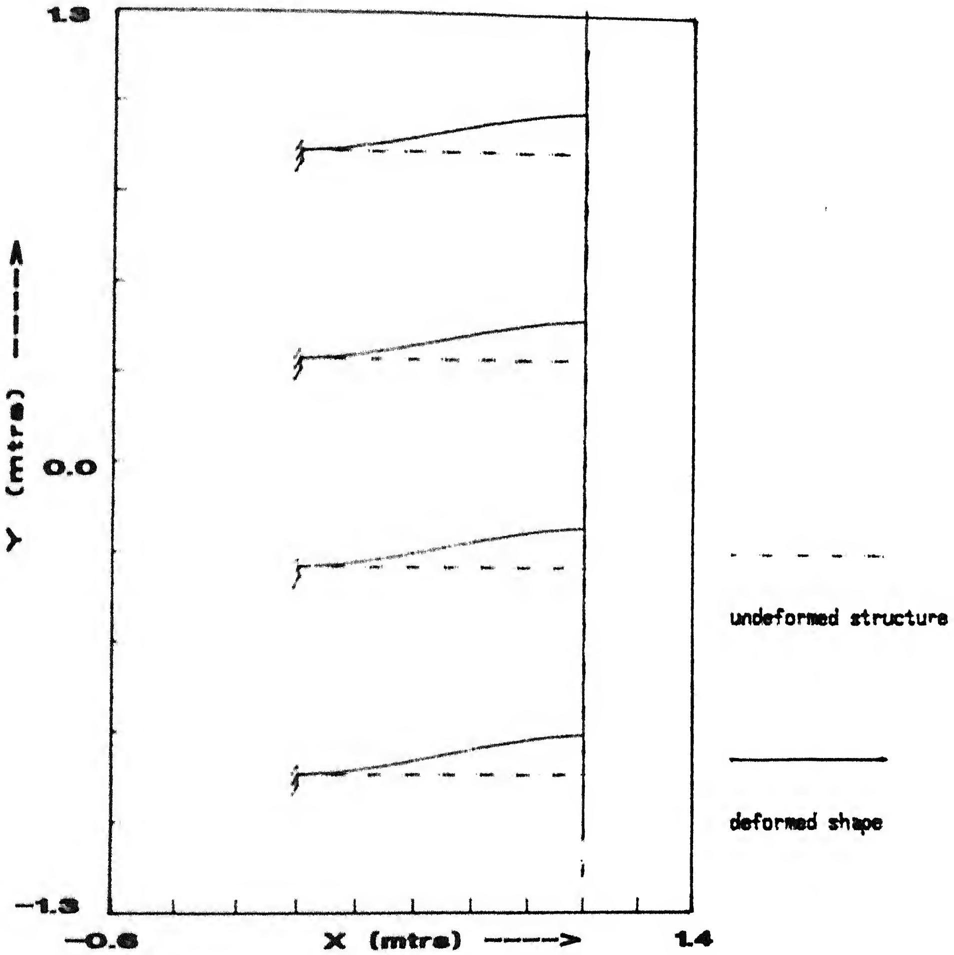


Fig.(3.10) :Mode Shape for Banded blade

3rd Mode (n=2)

(w=0.31211831 E+04) Rad/sec

Solution Method :Substructuring

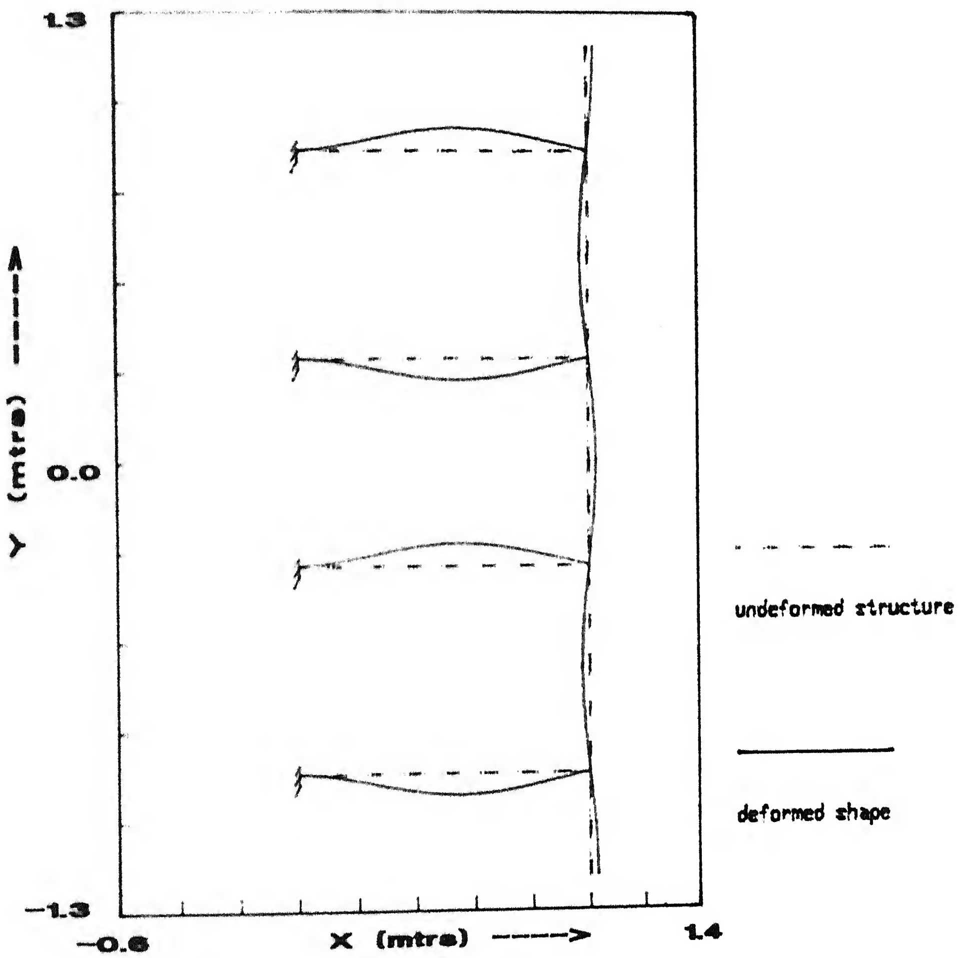


Fig(3.11) :Mode Shape for Banded Blade

1st Mode

(w= 0.01682748 E+04) Rad/sec

Solution Method :Complete FEM assembly

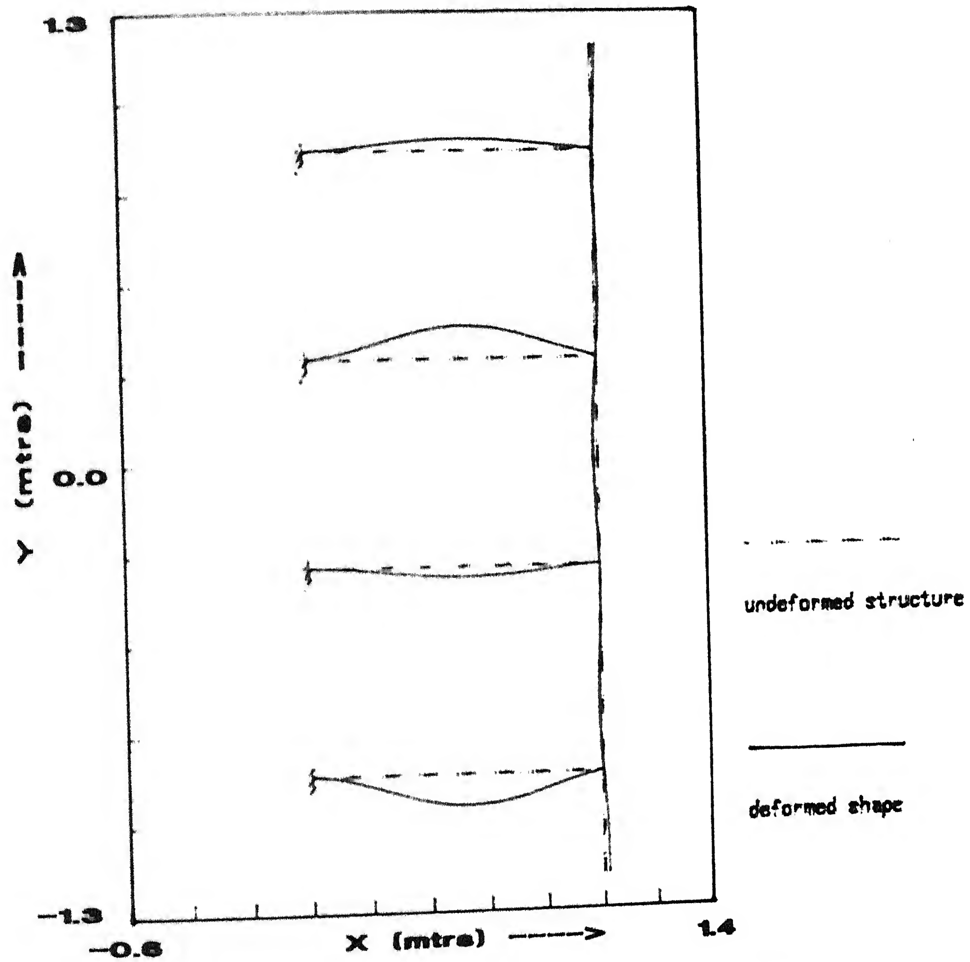


Fig(3.12) :Mode Shape for Banded Blade

2nd Mode

(w= 0.09159479 E+04) Rad/sec

Solution Method :Complete FEM assembly

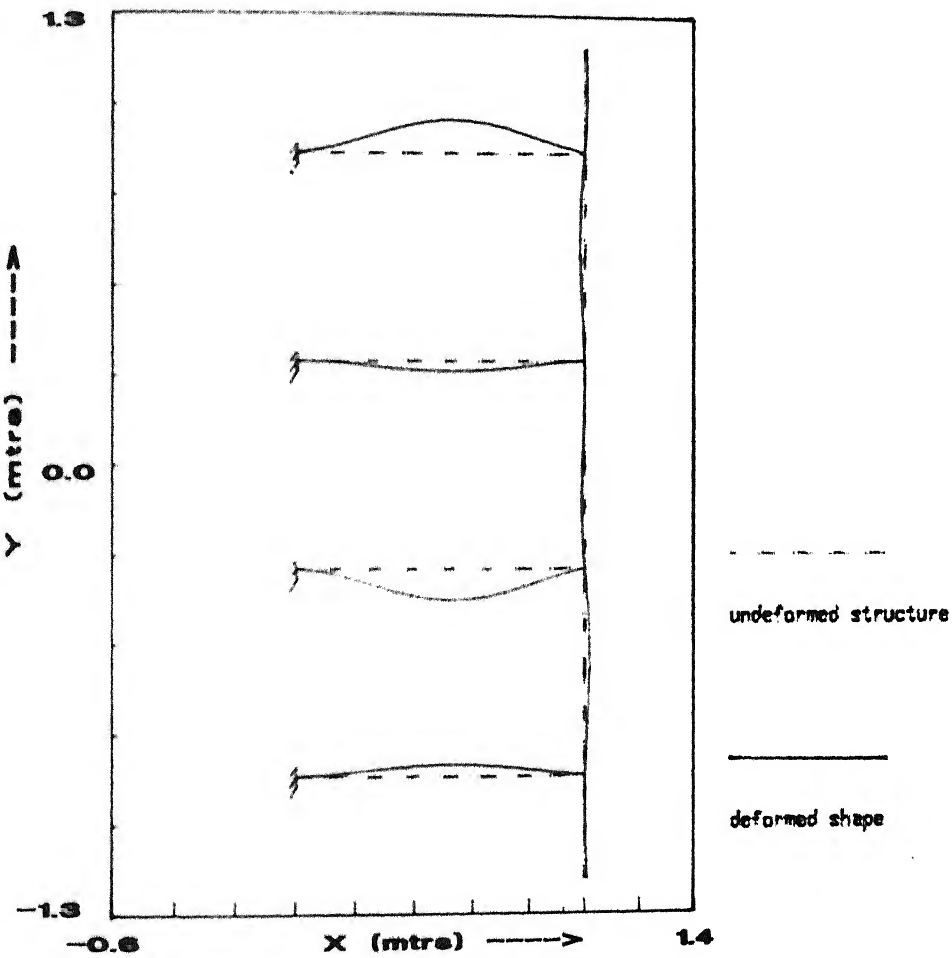


Fig(3.13) :Mode Shape for Banded Blade

3rd Mode

(w= 0.10035903 E+04) Rad/sec

Solution Method :Complete FEM assembly

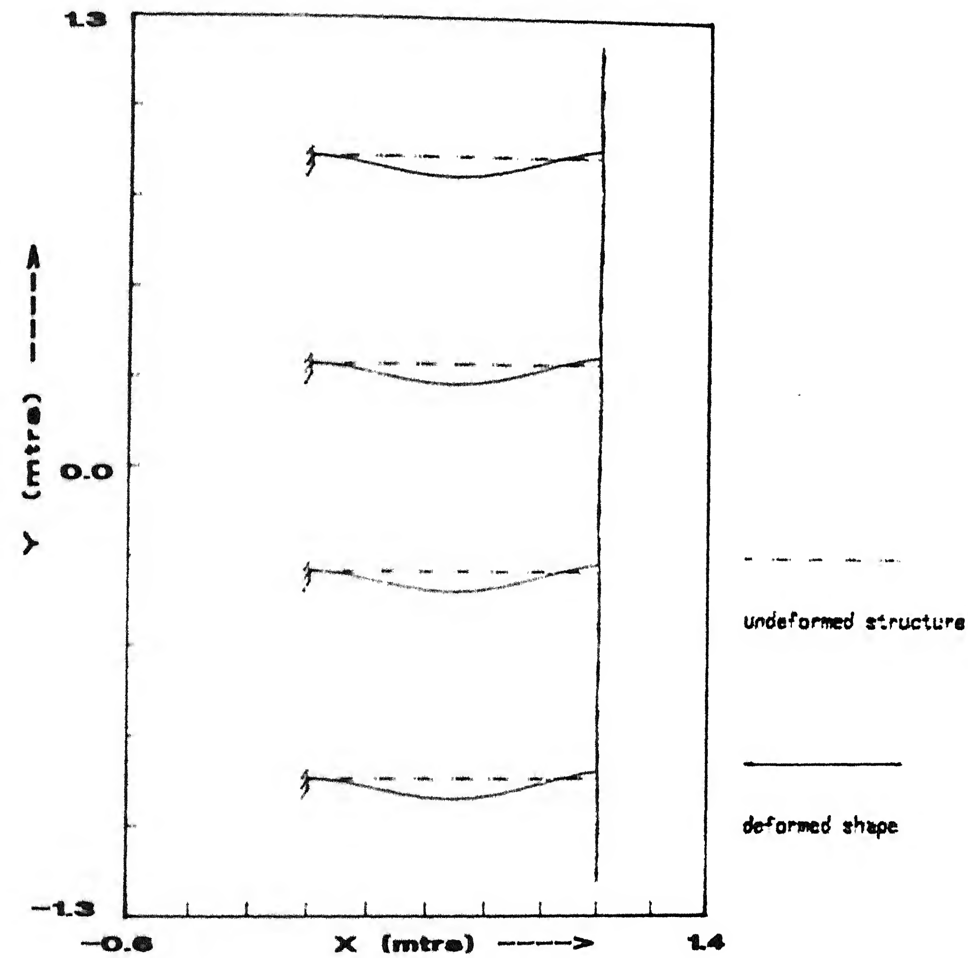


Fig(3.14) :Mode Shape for Banded Blade

4th Mode

(w= 0.10035903 E+04) Rad/sec

Solution Method :Complete FEM assembly

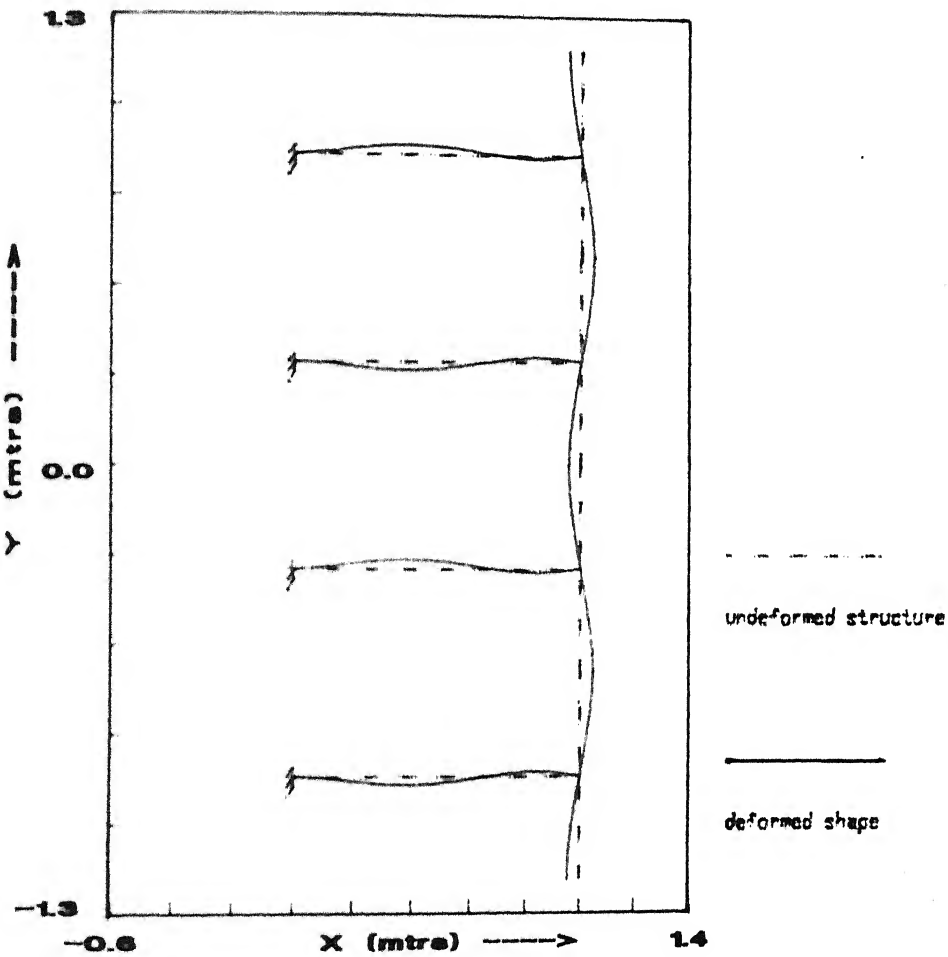


Fig(3.15) :Mode Shape for Banded Blade

5th Mode

(w= .11581150 E+04) Rad/sec

Solution Method :Complete FEM assembly

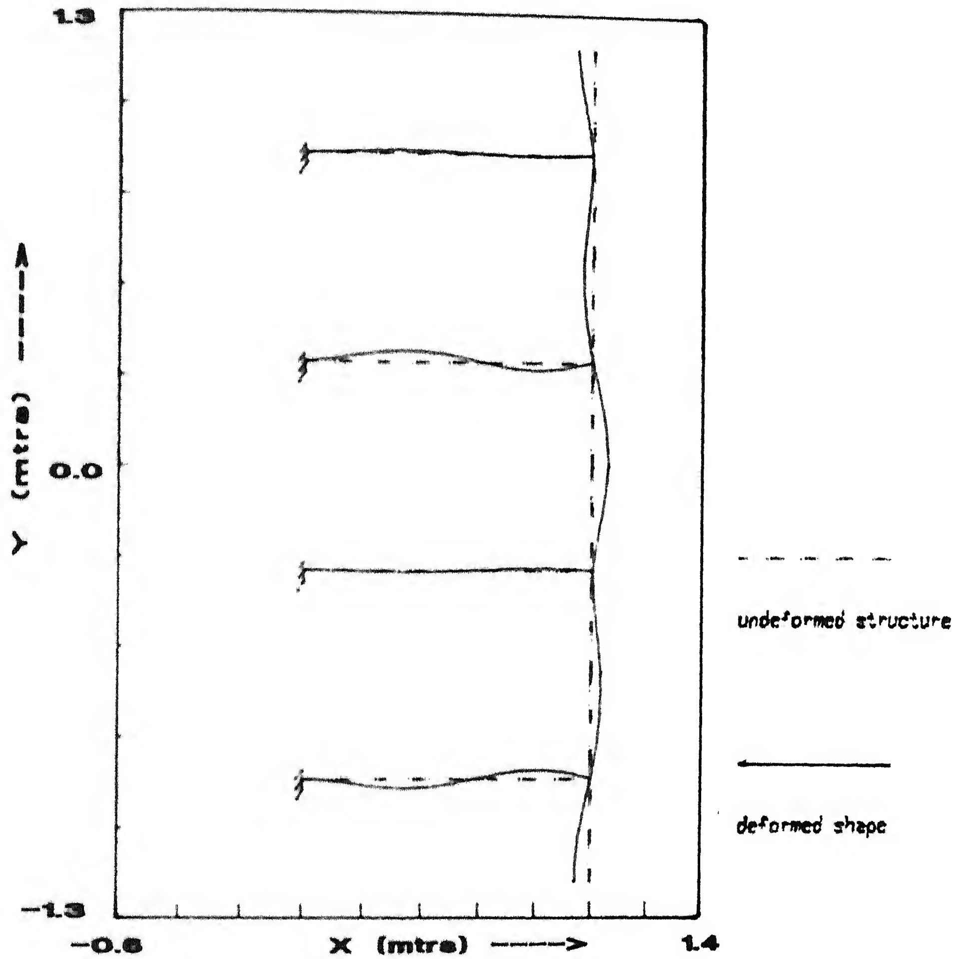


Fig(3.16) :Mode Shape for Banded Blade

6th Mode

(w= 0.18420031 E+04) Rad/sec

Solution Method :Complete FEM assembly

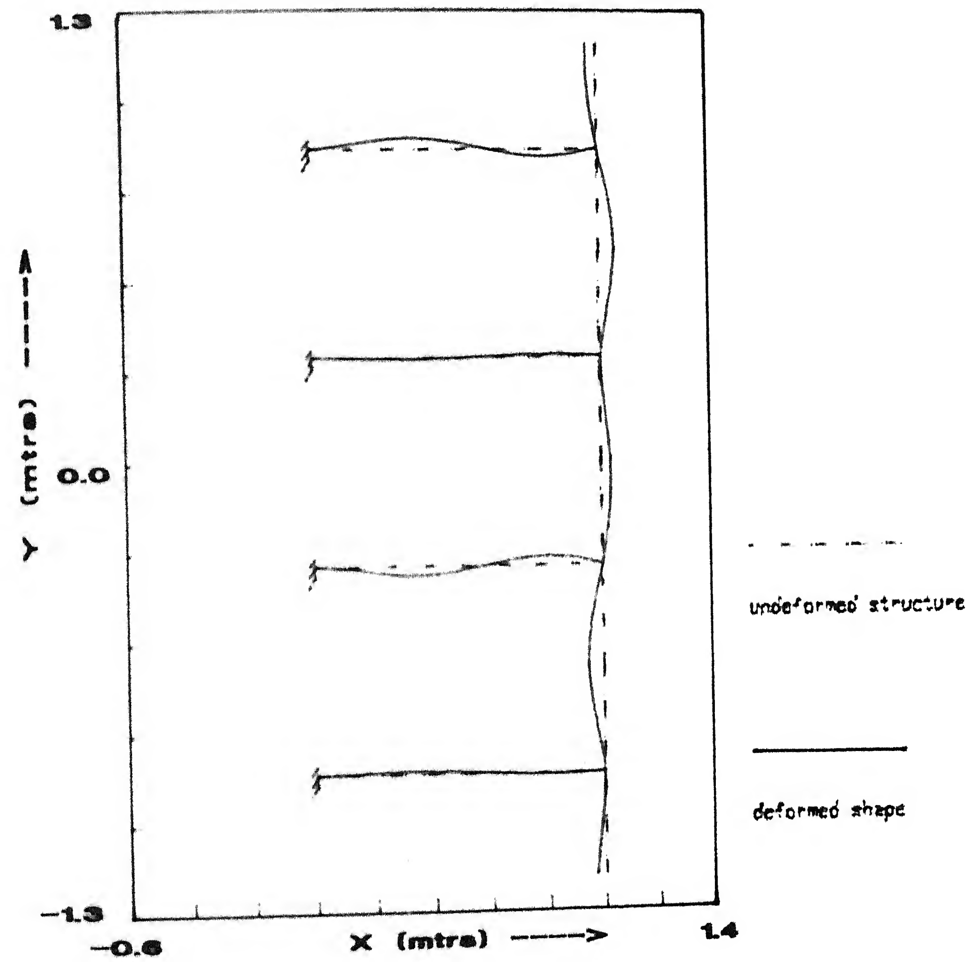


Fig(3.17) :Mode Shape for Banded Blade

7th Mode

(w= 0.22471453 E+04) Rad/sec

Solution Method :Complete FEM assembly

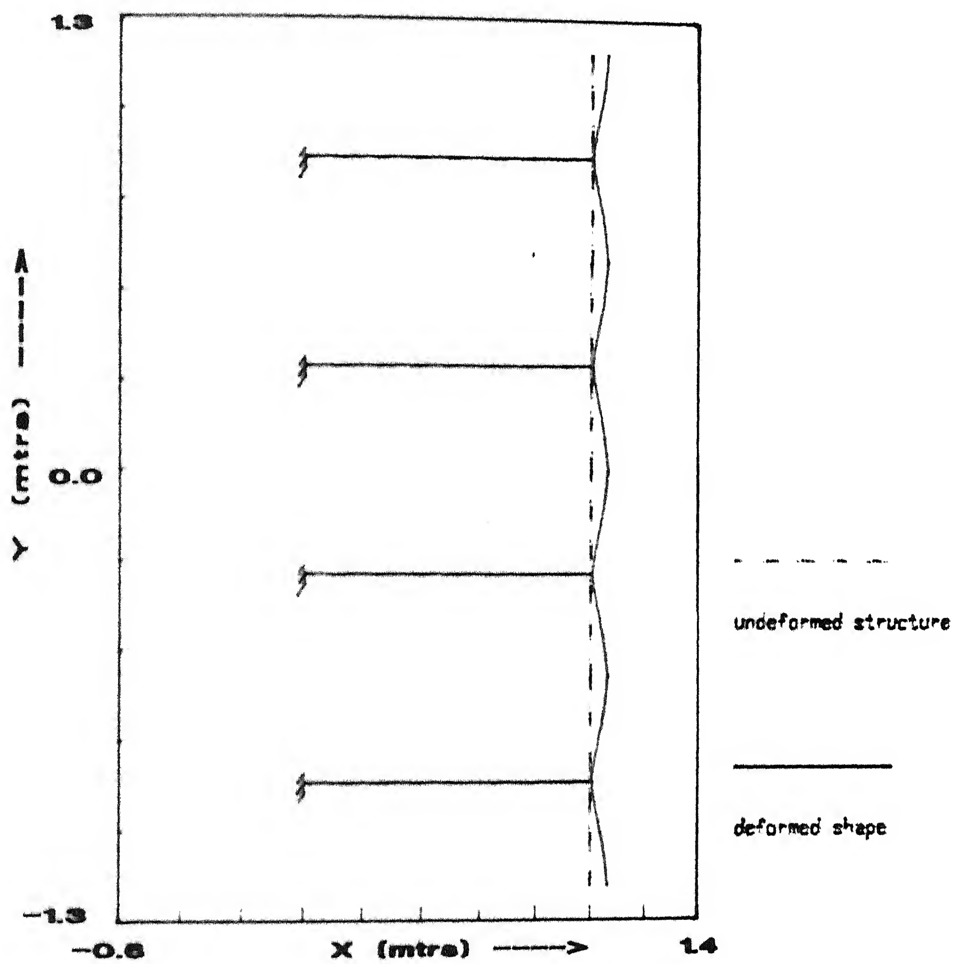


Fig(3.18) :Mode Shape for Banded Blade

8th Mode

(w= 0.2247145З E+04) Rad/sec

Solution Method :Complete FEM assembly

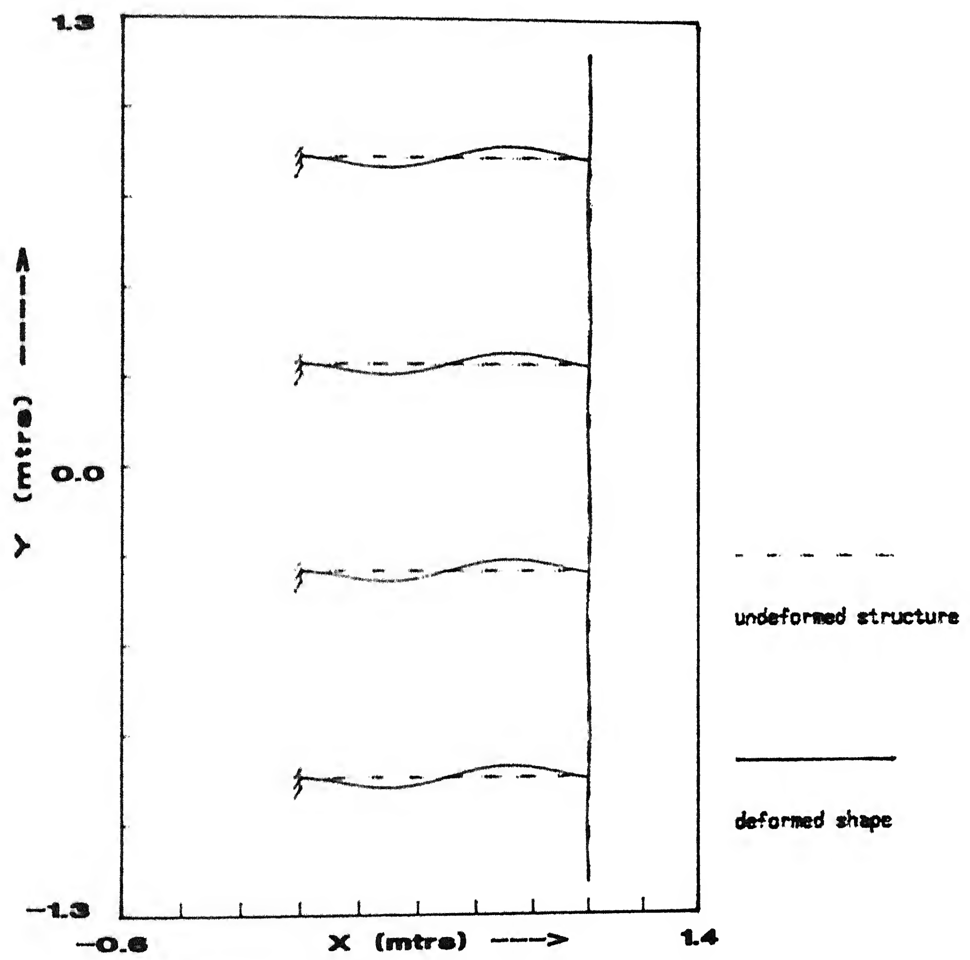


Fig(3.19) :Mode Shape for Banded Blade

9th Mode

(w= 0.28817944 E+04) Rad/sec

Solution Method :Complete FEM assembly

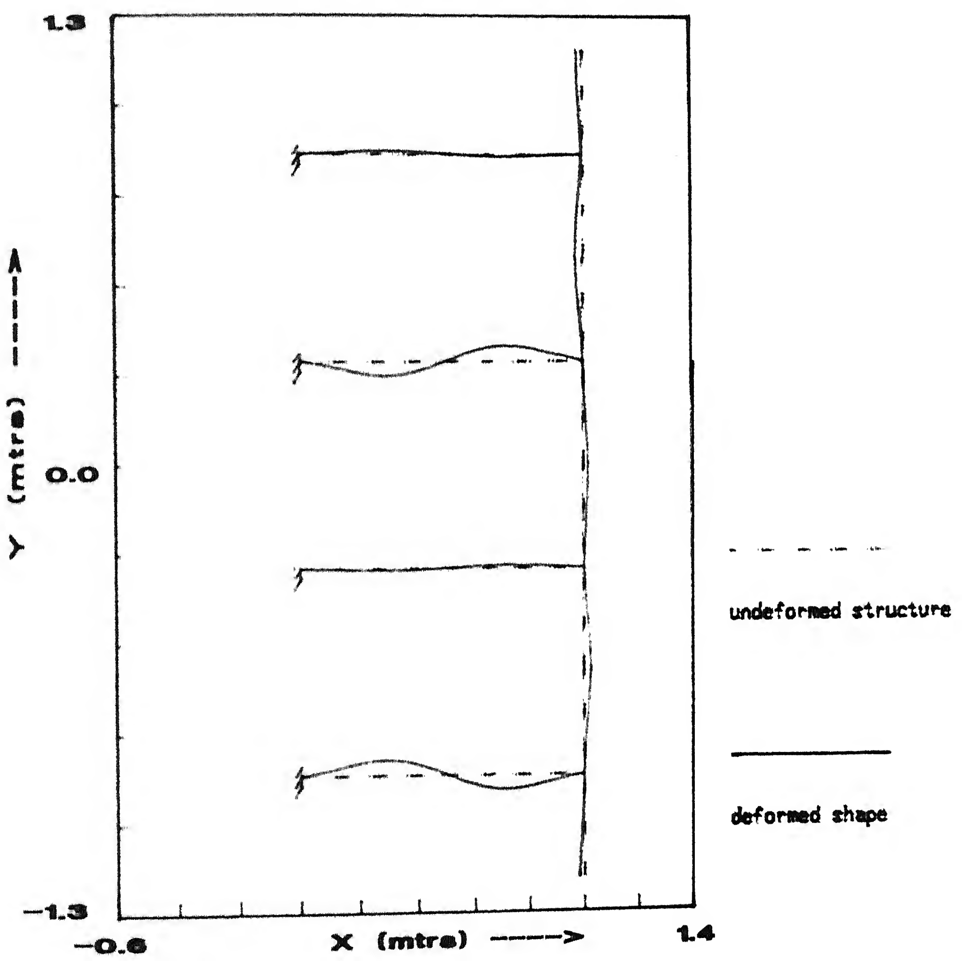


Fig(3.20) :Mode Shape for Banded Blade

10th Mode

(w= 0.30071595 E+04) Rad/sec

Solution Method :Complete FEM assembly

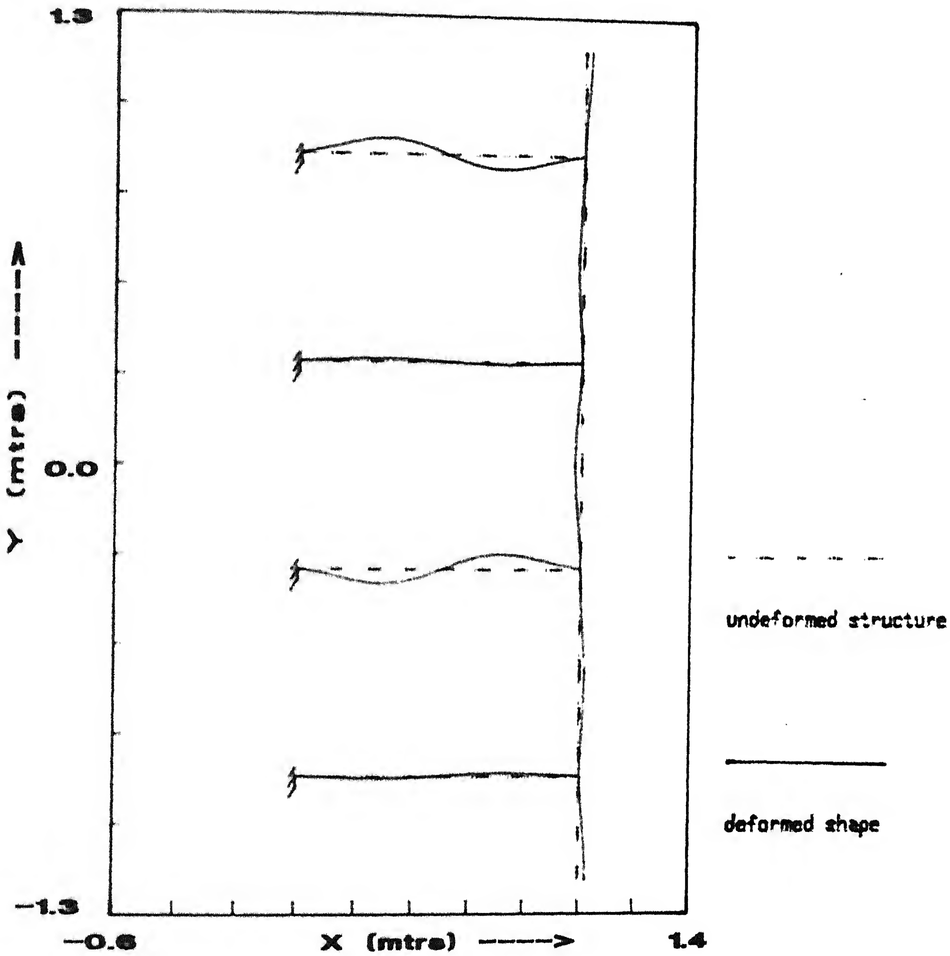


Fig(3.21) :Mode Shape for Banded Blade

11th Mode

(w= 0.30865460 E+04) Rad/sec

Solution Method :Complete FEM assembly

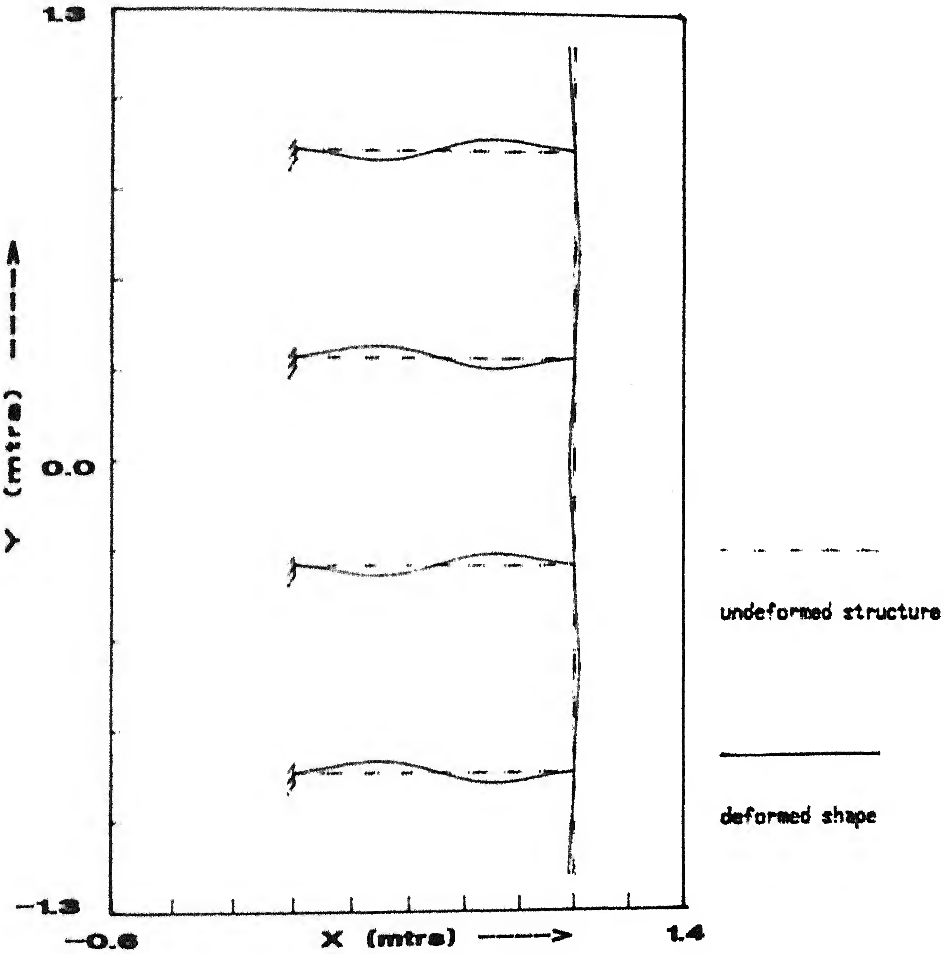


Fig(3.22) :Mode Shape for Banded Blade

12th Mode

(w= 0.30865460 E+04) Rad/sec

Solution Method :Complete FEM assembly



Fig(3.23) :Mode Shape for Banded Blade

13th Mode

(w= 0.31221229 E+04) Rad/sec

Solution Method :Complete FEM assembly

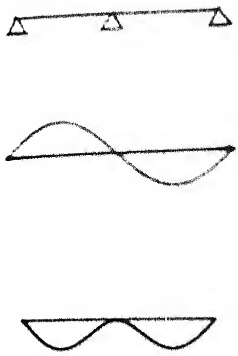
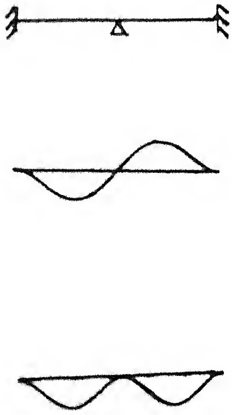


Fig.(3.24): Beam HHH and its first two modes



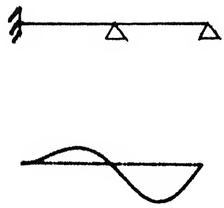


Fig.(3.25): Beam FHF and its first two modes

original structure

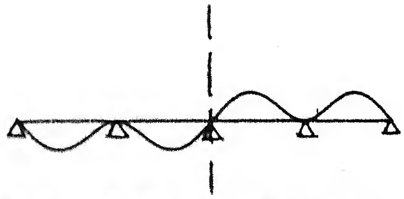


fig (3.27)equivalent cyclic periodic structure

original structure

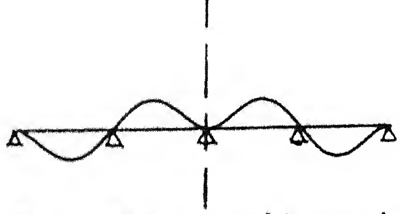
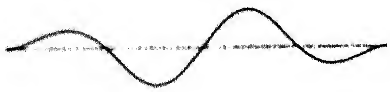


fig (3.28)equivalent cyclic periodic strucutre

original structure



intermediate structure

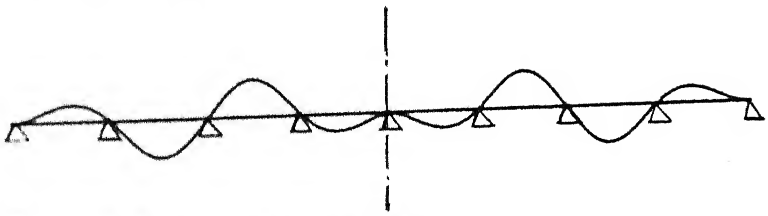


fig (3.29) equivalent cyclic periodic structure

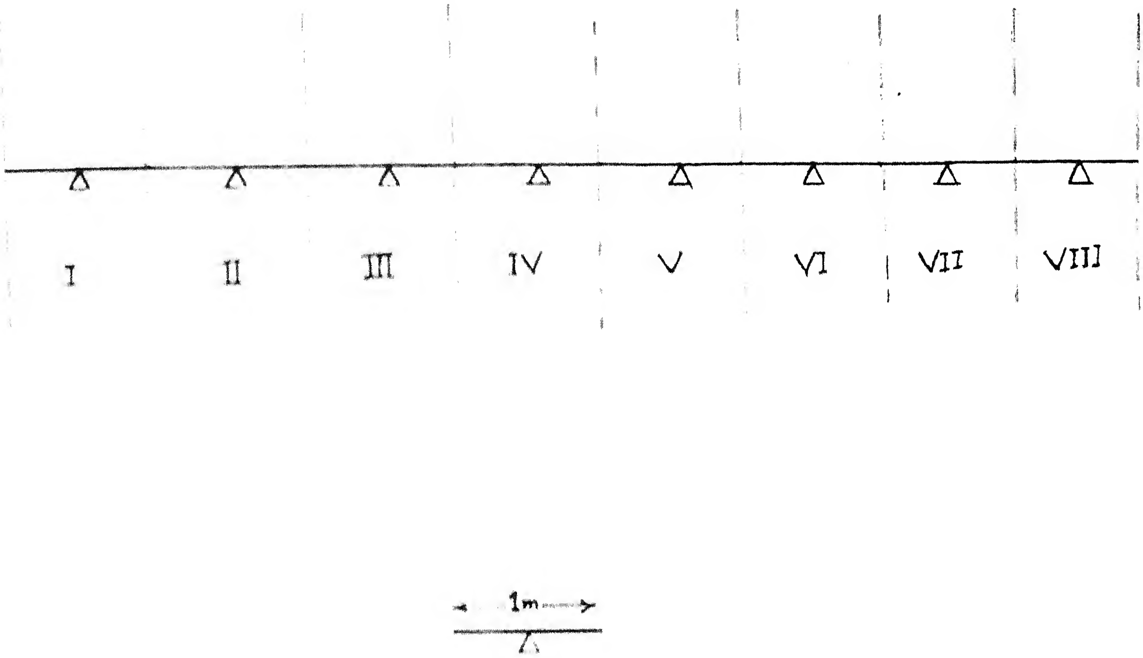


fig 3.30 substructure selected in problem 2

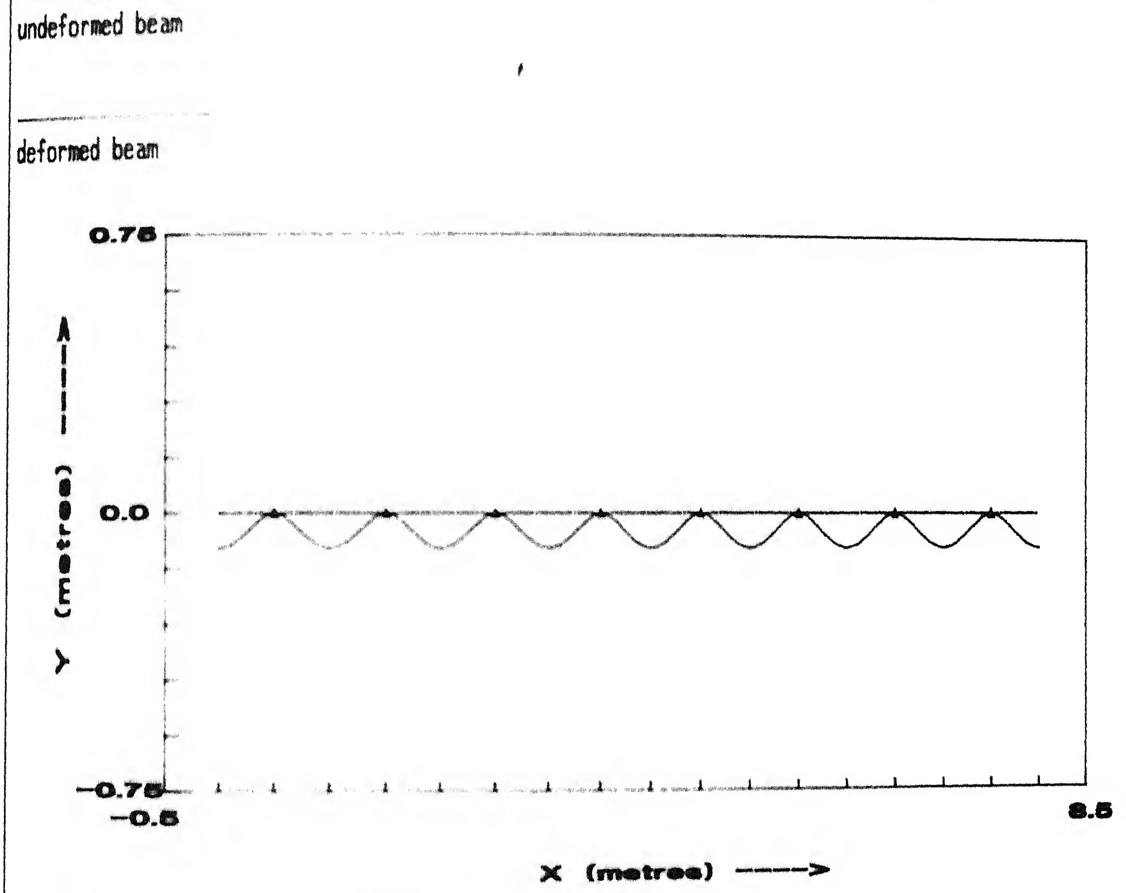


Fig. (3.31):Mode Shape for continuous beam

1st Mode (n=0)

w= 1132.9165 rad/s

(Equivalent Structure)

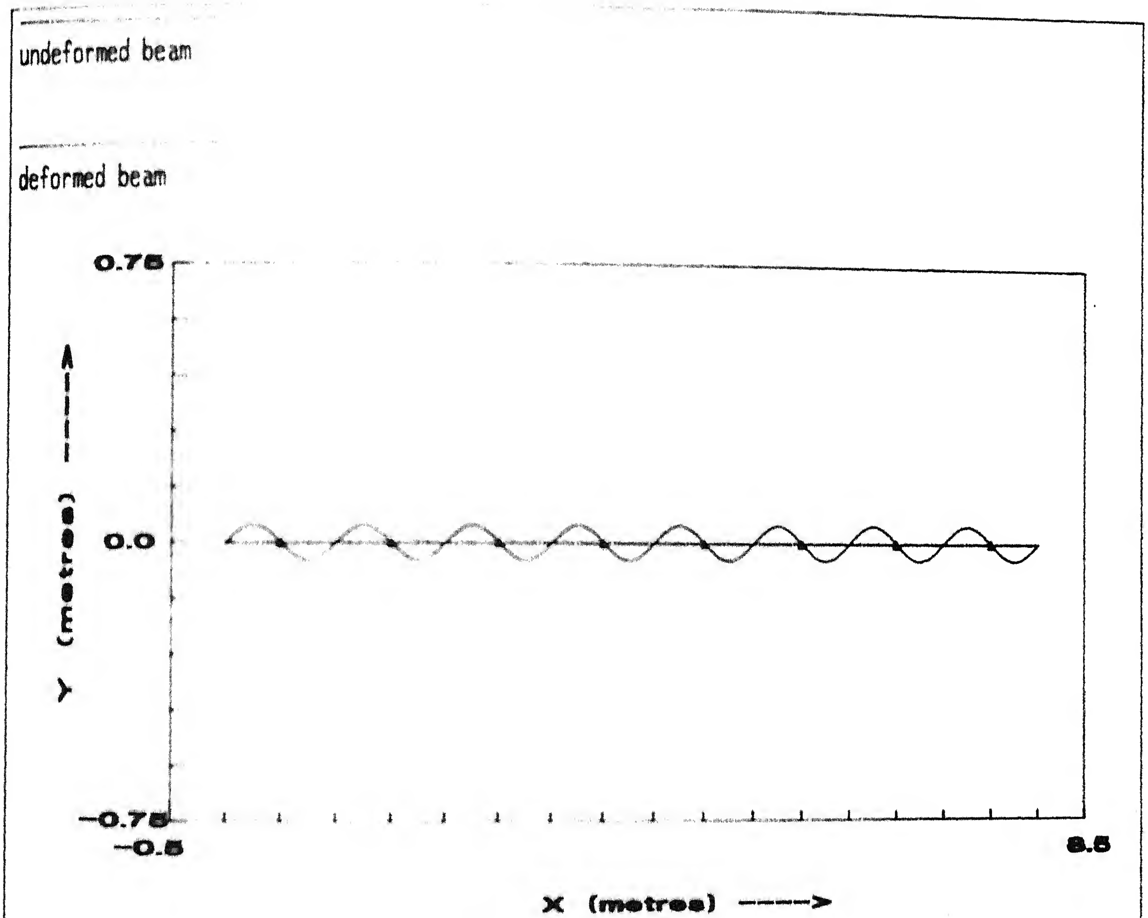


Fig. (3.32):Mode Shape for continuous beam

2nd Mode (n=0)

w= 1999.0738 rad/s

(Equivalent Structure)

undeformed beam

deformed beam

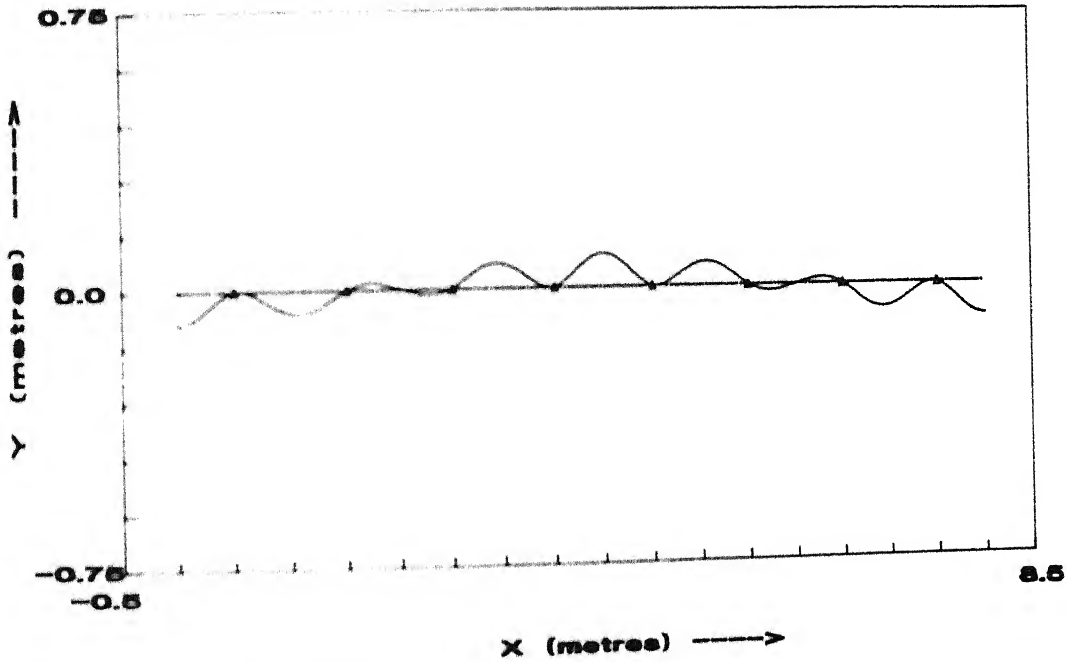


Fig. (3.33a):Mode Shape for continuous beam

1st Mode (n=1) vector {u}

w= 1008.7532 rad/s

(Equivalent Structure)

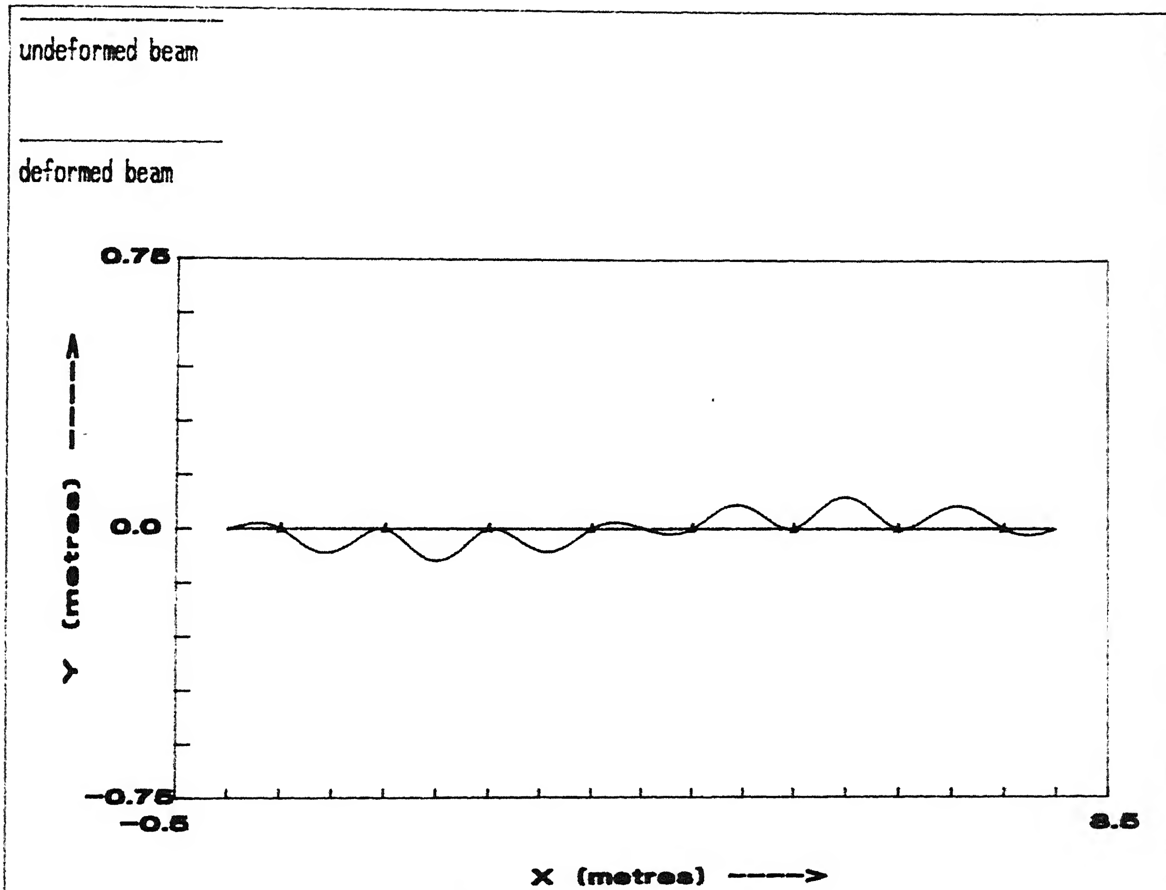


Fig. (3.33b):Mode Shape for continuous beam

1st Mode (n=1) vector $\{\overline{u}\}$

w= 1008.7532 rad/s

(Equivalent Structure)

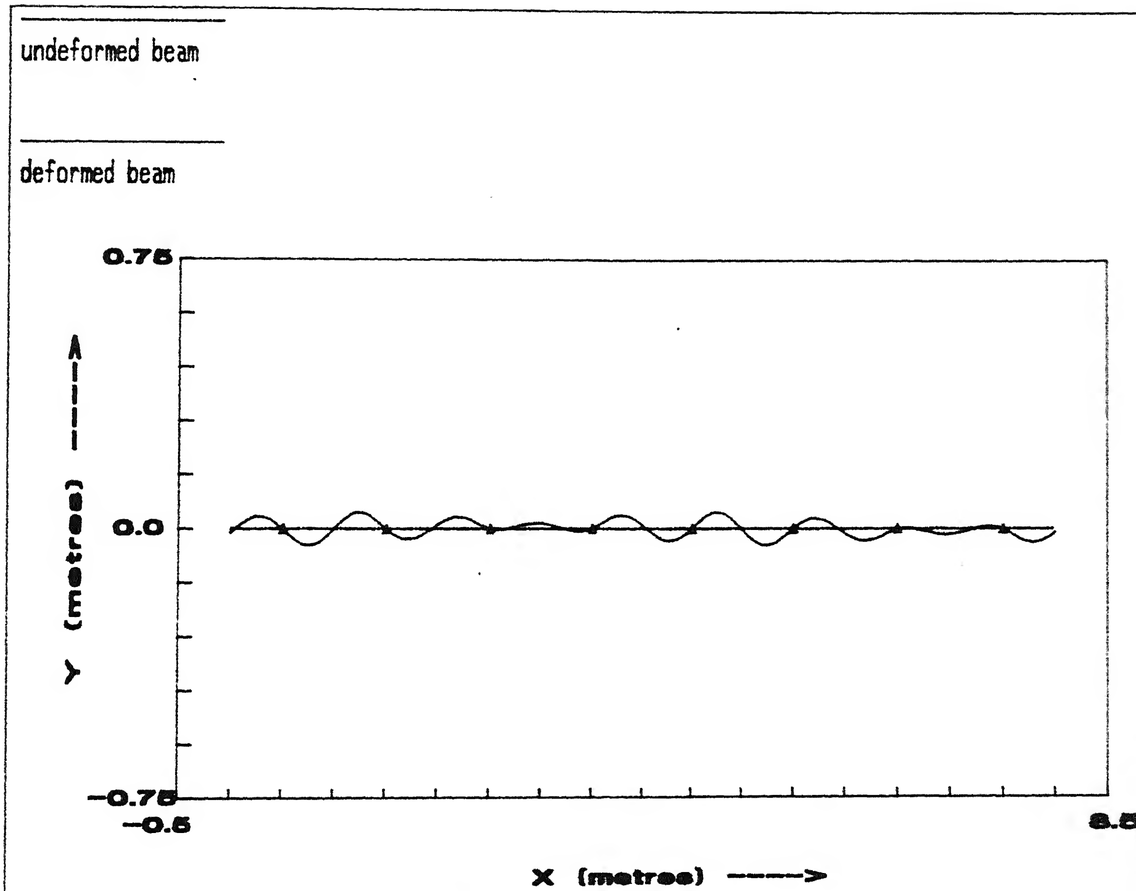


Fig. (3.34a):Mode Shape for continuous beam

2nd Mode (n=1) vector {u}

w= 2169.4196 rad/s

(Equivalent Structure)

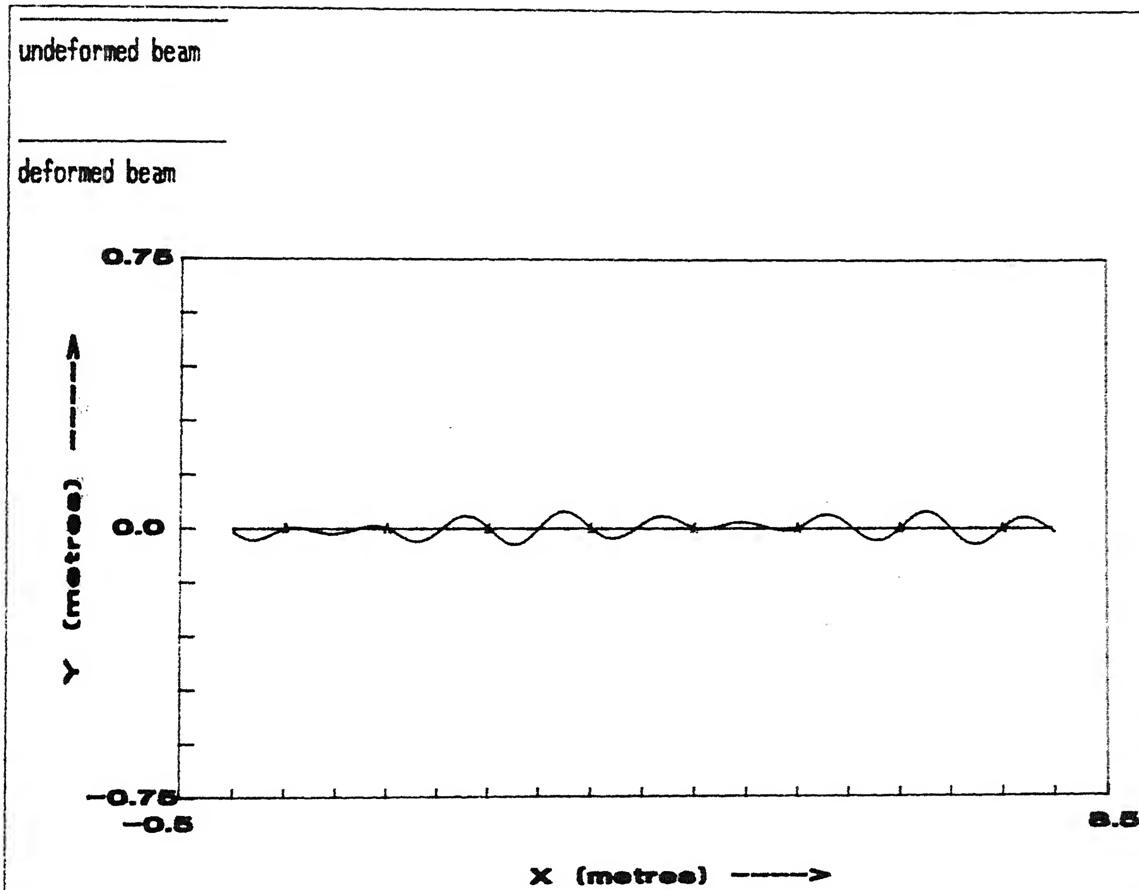


Fig. (3.34b):Mode Shape for continuous beam

2nd Mode (n=1) vector $\{\bar{u}\}$

w= 2169.4196 rad/s

(Equivalent Structure)

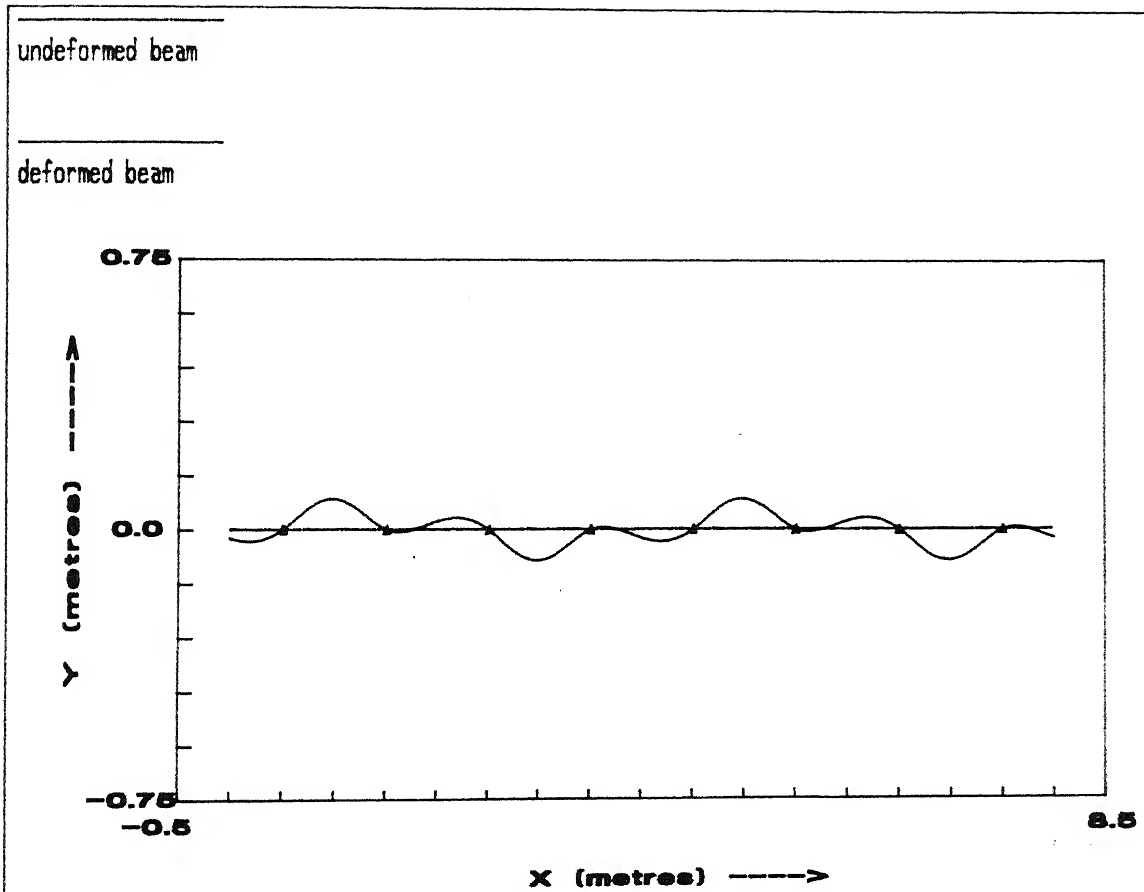


Fig. (3.35a):Mode Shape for continuous beam

1st Mode (n=2) vector {u}

w= 780.7320 rad/s

(Equivalent Structure)

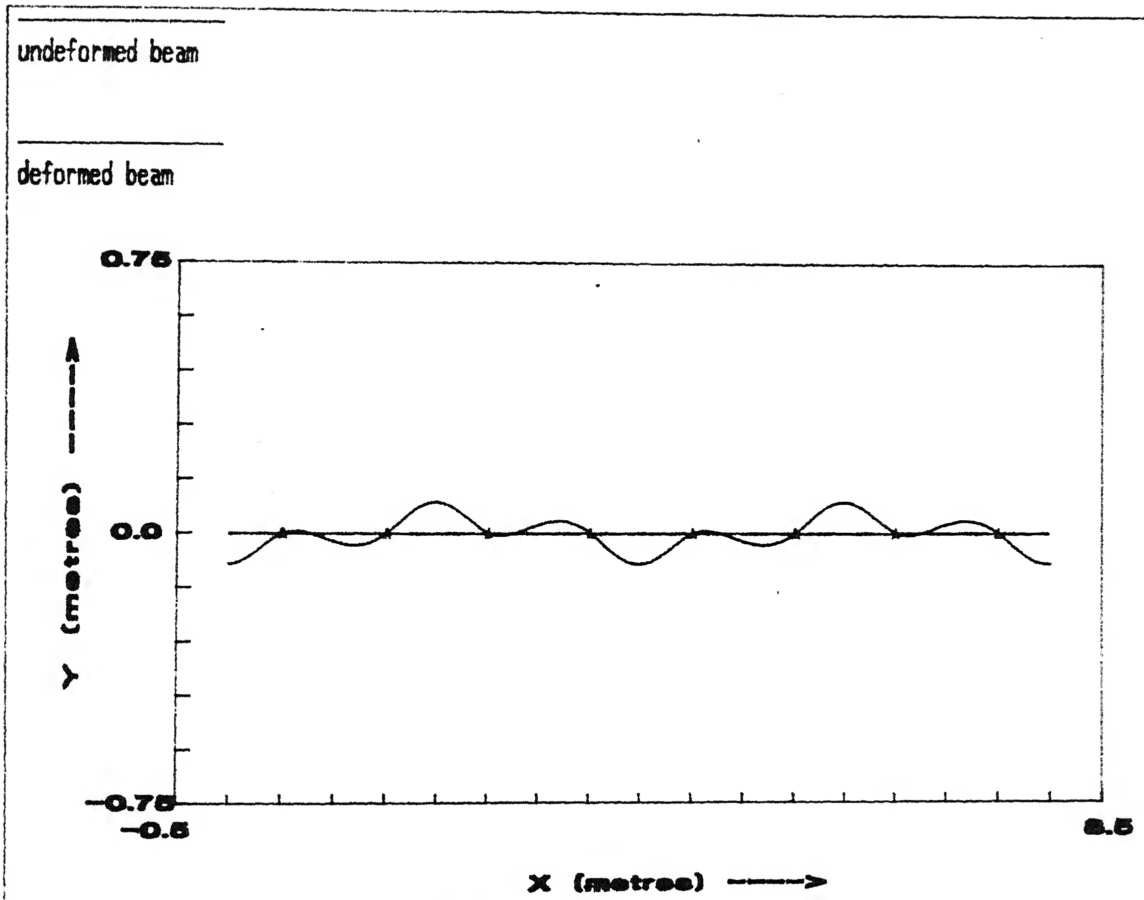


Fig. (3.35b):Mode Shape for continuous beam

1st Mode (n=2) vector $\{\bar{u}\}$

w= 780.7320 rad/s

(Equivalent Structure)

undeformed beam

deformed beam

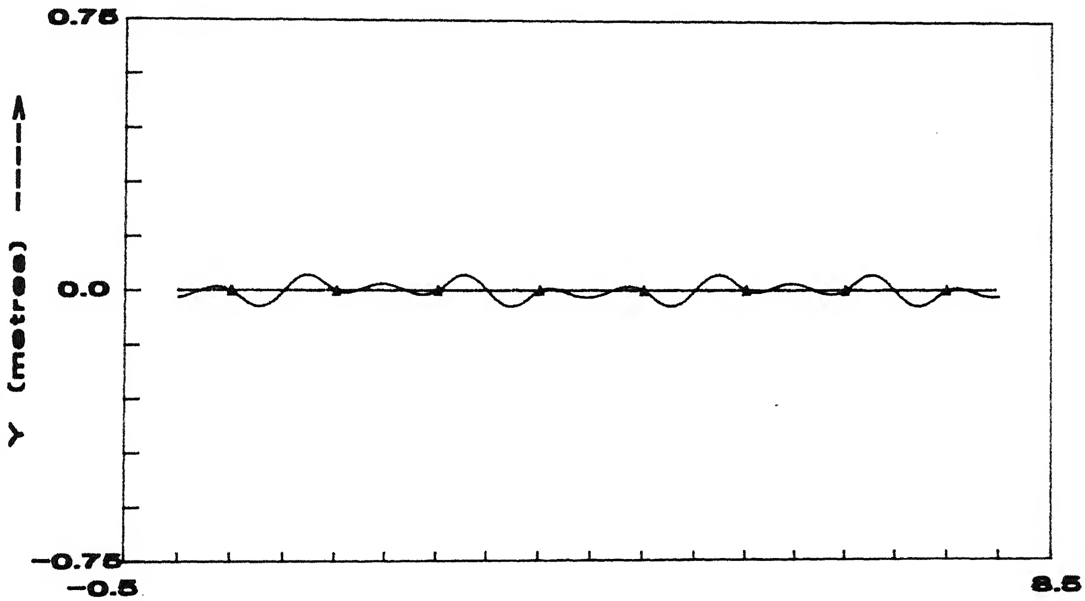


Fig. (3.36a):Mode Shape for continuous beam

2nd Mode (n=2) vector {u}

w= 2530.0820 rad/s

(Equivalent Structure)

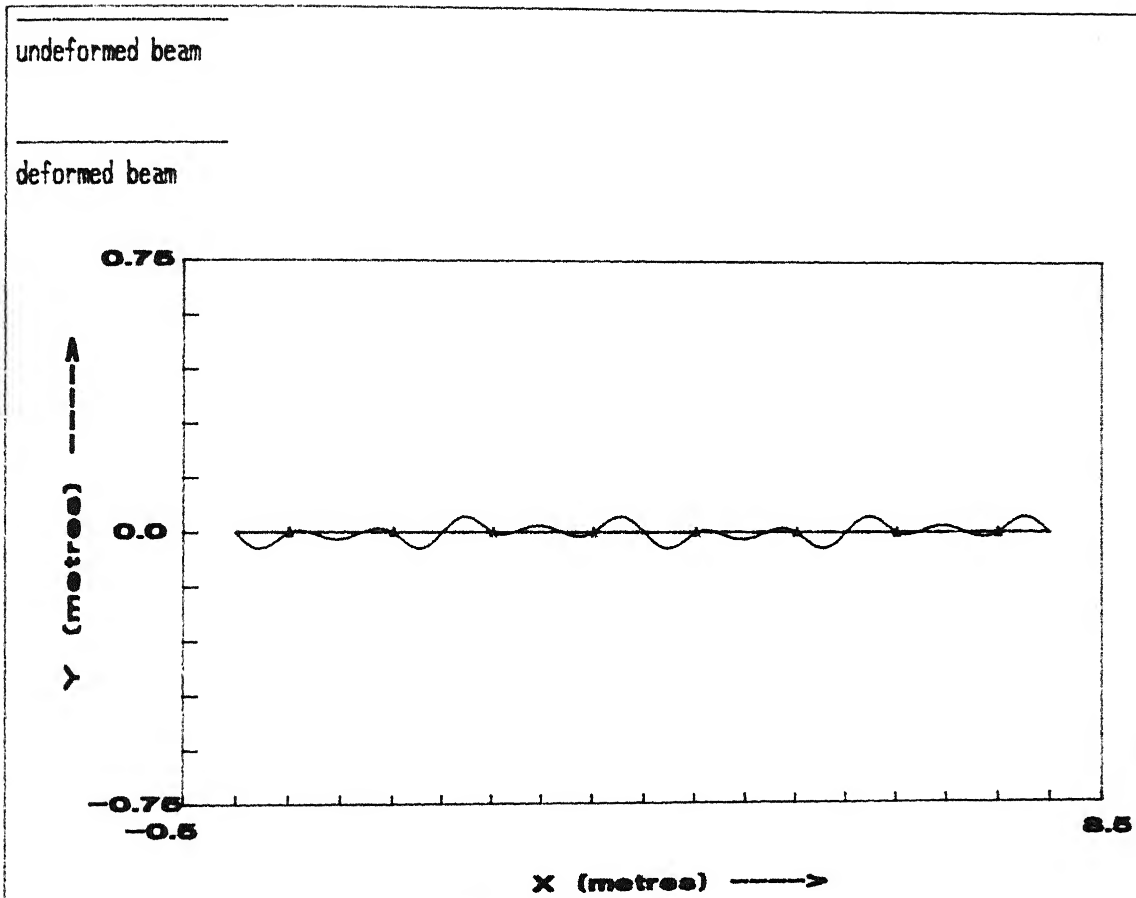


Fig. (3.36b):Mode Shape for continuous beam

2nd Mode (n=2) vector $\{\bar{u}\}$

w= 2530.0920 rad/s

(Equivalent Structure)

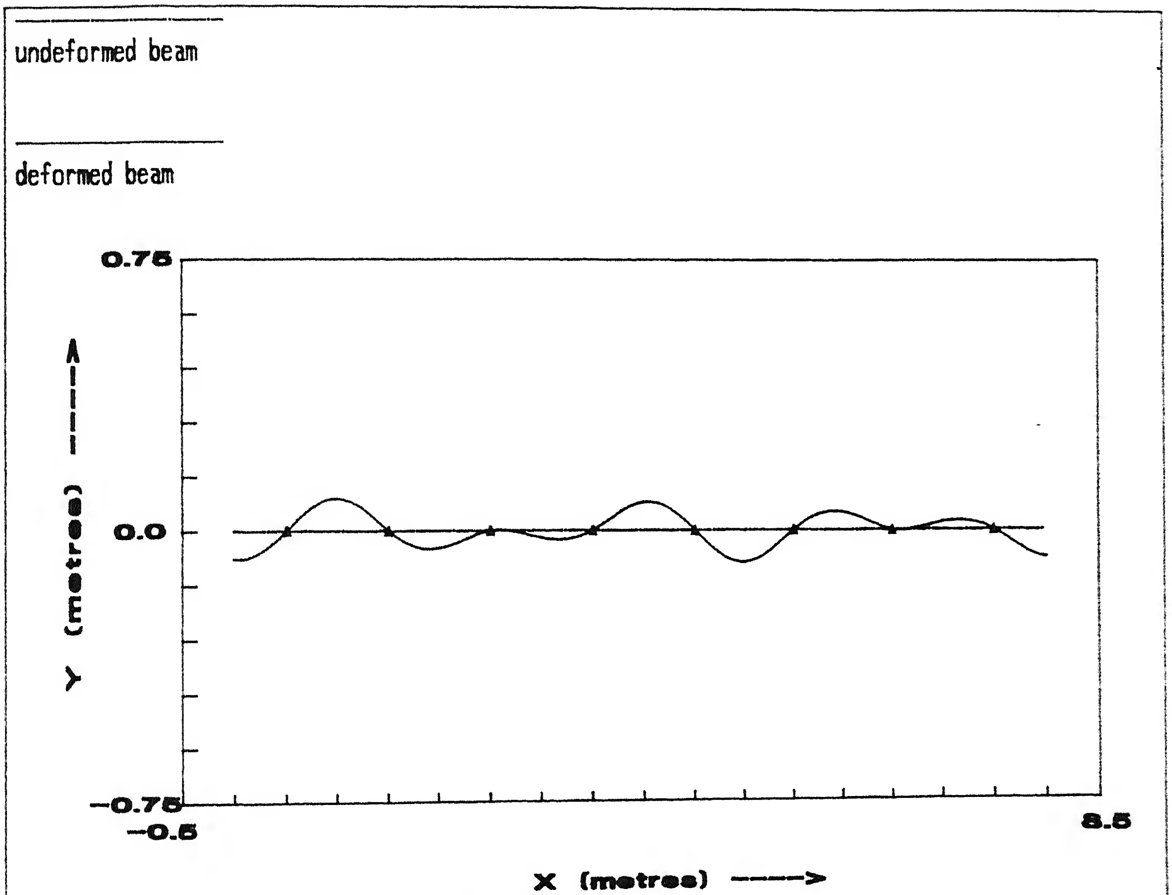


Fig. (3.37a):Mode Shape for continuous beam

1st Mode (n=3) vector {u}

w= 583.0348 rad/s

(Equivalent Structure)

undeformed beam

deformed beam

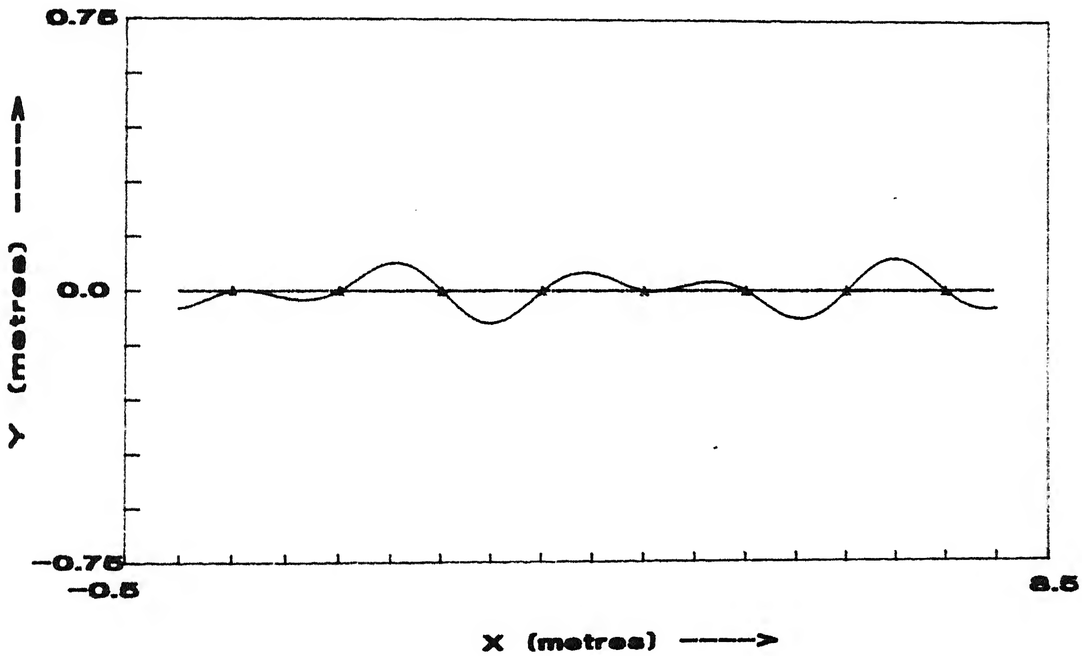


Fig. (3.37b):Mode Shape for continuous beam

1st Mode (n=3) vector $\{\bar{u}\}$

w= 583.0348 rad/s

(Equivalent Structure)

undeformed beam

deformed beam

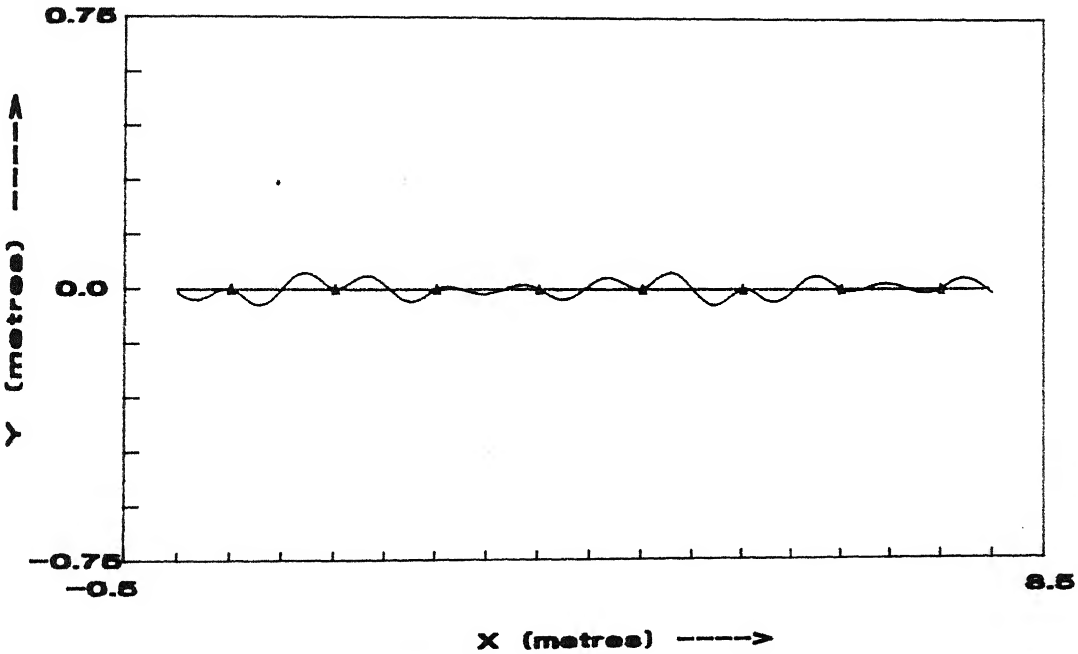


Fig. (3.38 a):Mode Shape for continuous beam

2nd Mode (n=3) vector {u}

w= 2918.3645 rad/s

(Equivalent Structure)

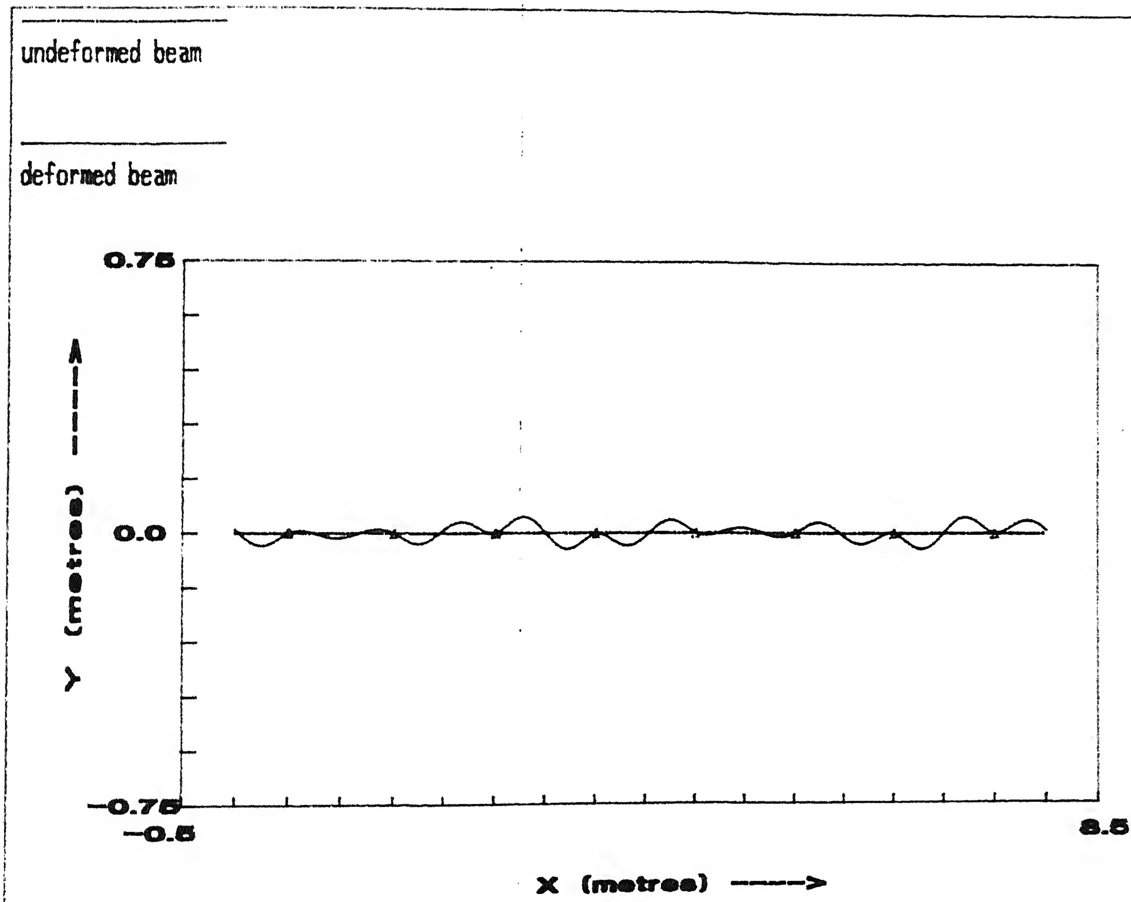


Fig. (3.38 b):Mode Shape for continuous beam

2nd Mode (n=3) vector $\{\bar{u}\}$

w= 2918.3645 rad/s

(Equivalent Structure)

undeformed beam

deformed beam

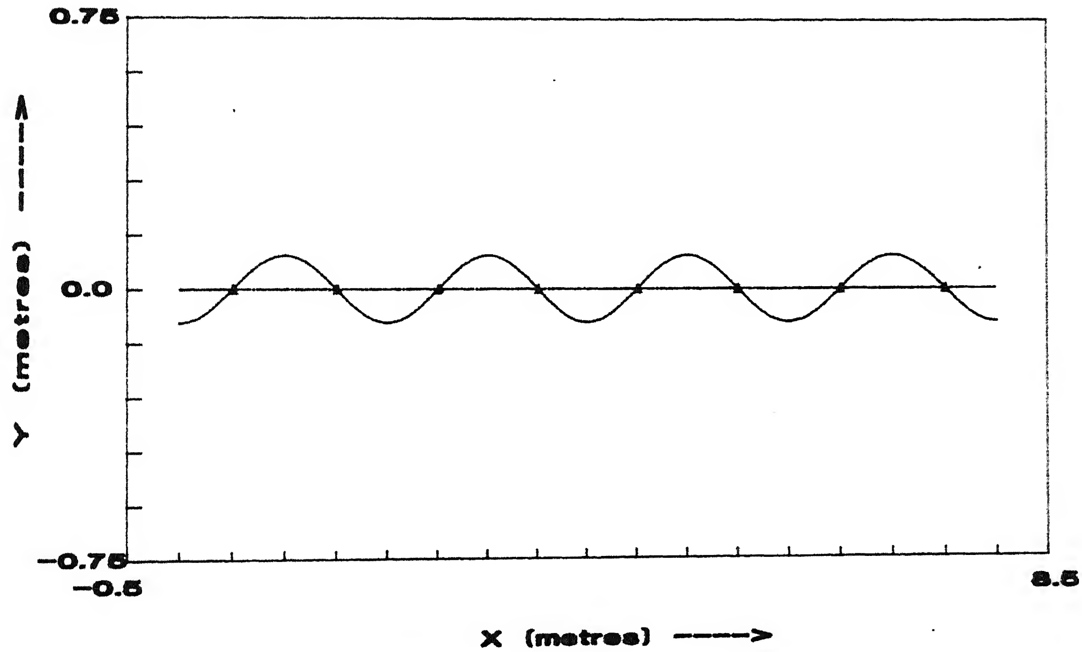


Fig. (3.39):Mode Shape for continuous beam

1st Mode (n=4)

w= 499.1668 rad/s

(Equivalent Structure)

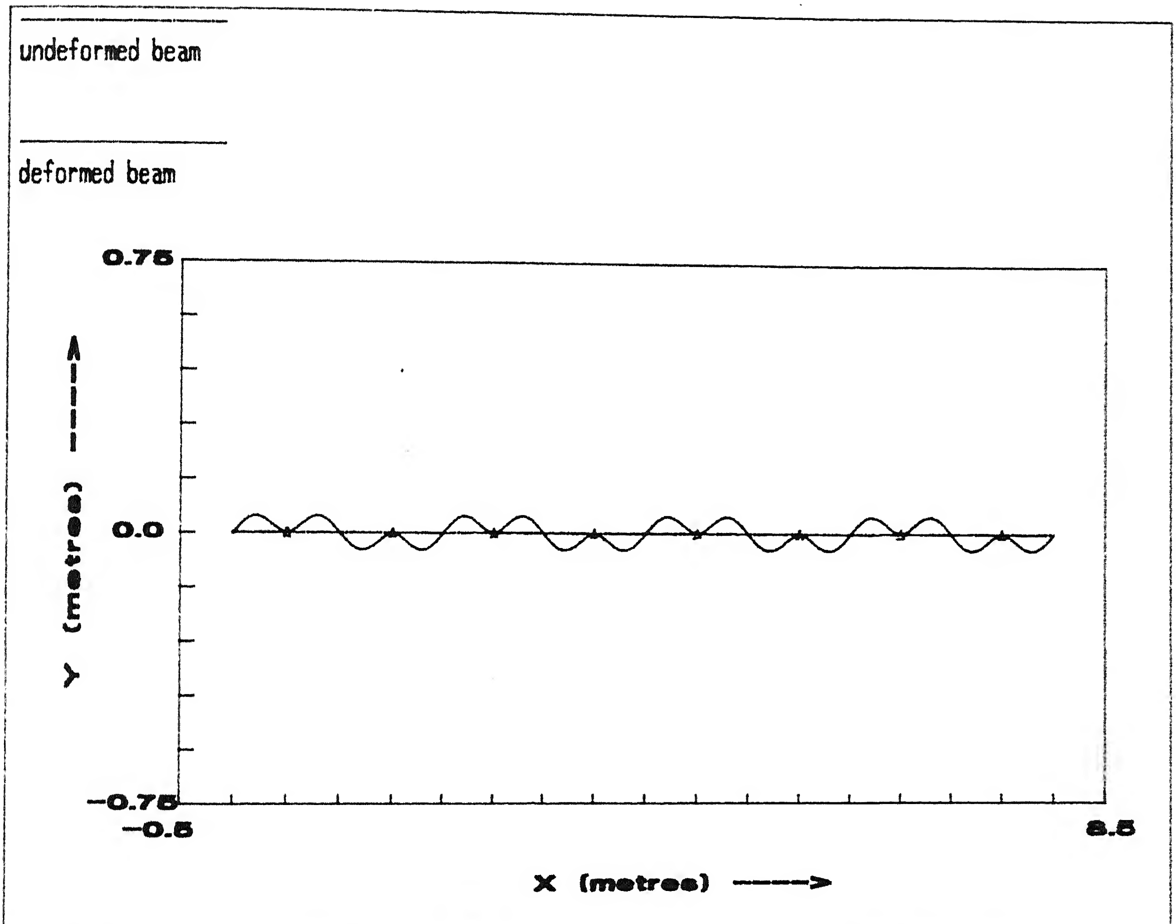


Fig. (3.40):Mode Shape for continuous beam

2nd Mode (n=4)

w= 3122.9494 rad/s

(Equivalent Structure)

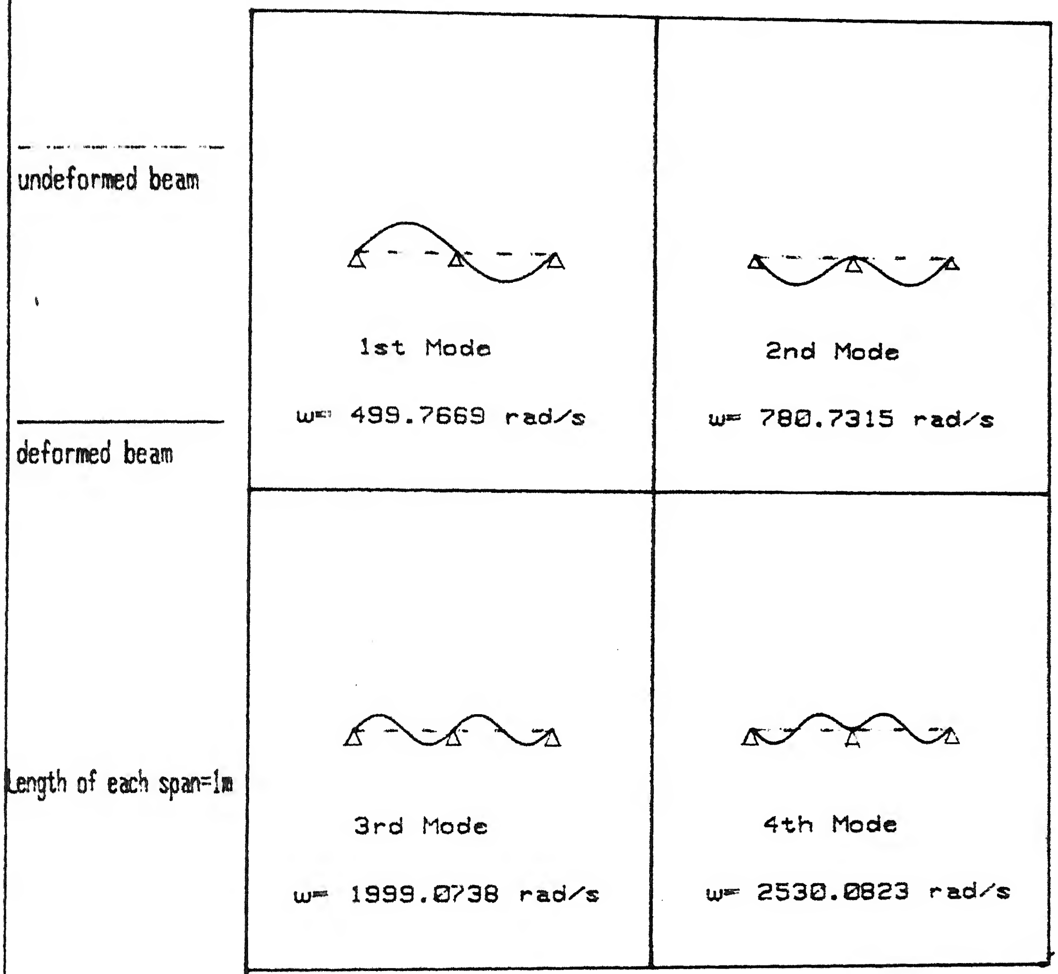


Fig.(3.41) :Mode Shapes For 2 span Beam

(Beam type-HHH)

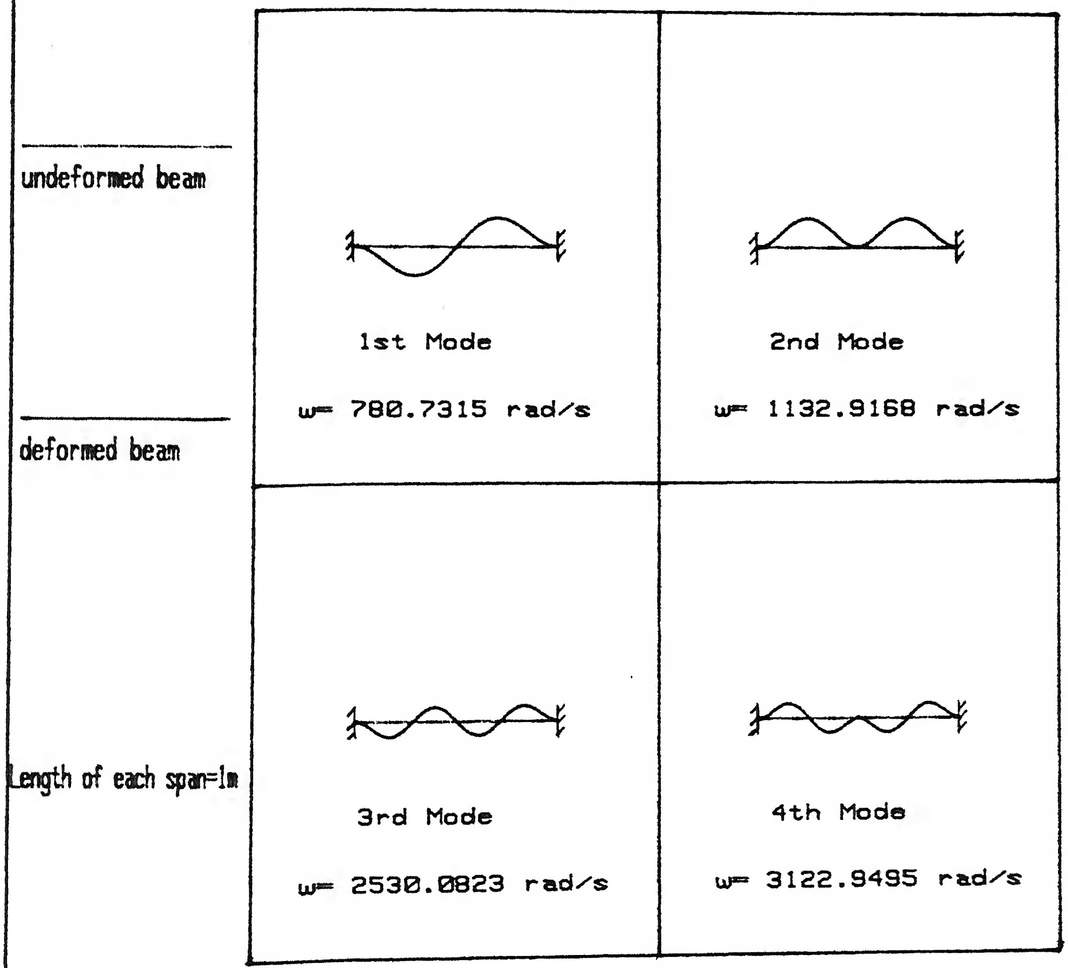


Fig.(3.42) :Mode Shapes For 2 span Beam

(Beam type-FHF)

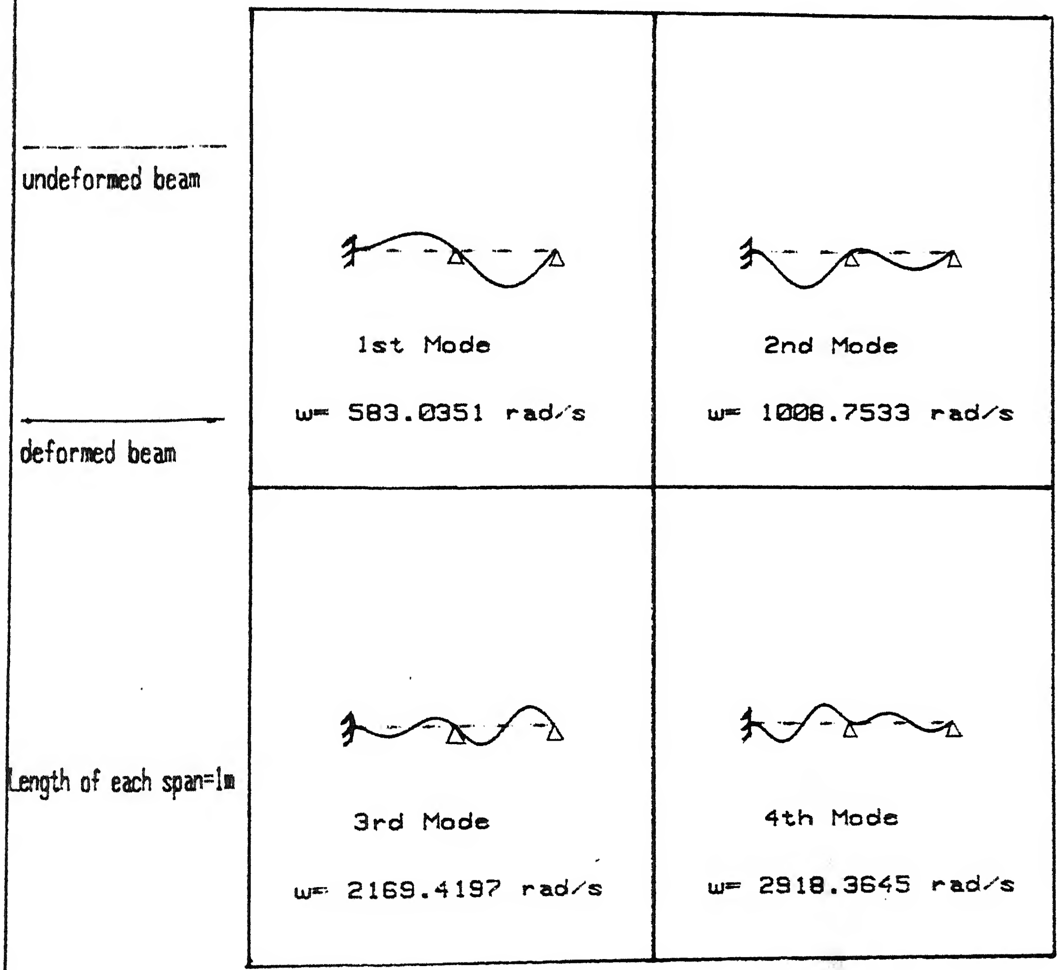


Fig.(3.43) :Mode Shapes For 2 span Beam

(Beam type-FHH)

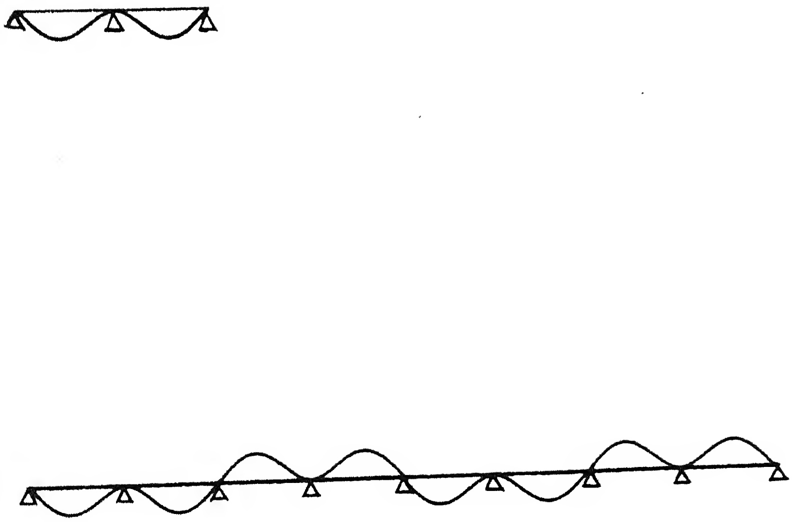


Fig.(3.44a): Second mode of beam HHH extended over eight spans

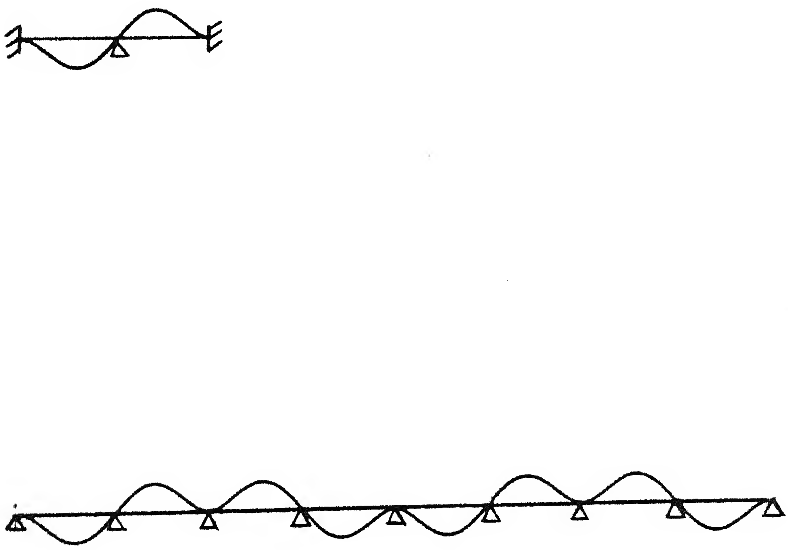


fig (3.44b): 1st mode of two-sapn beam (type-FHF) extended over eight spans

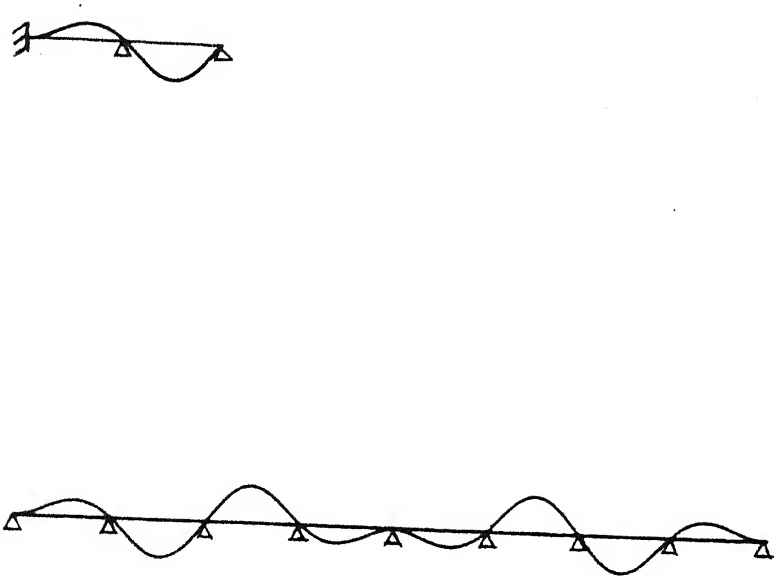


Fig. (3.45): First mode of Beam FHH extended over eight spans

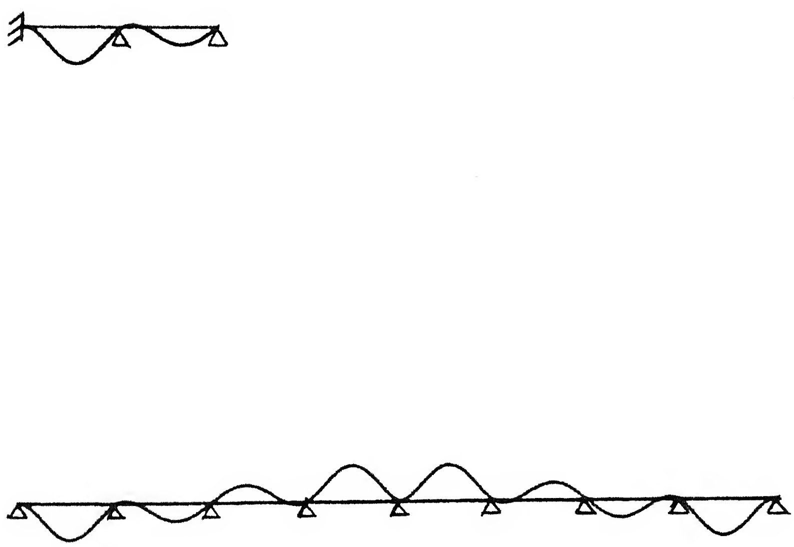


Fig. (3.46): Second mode of beam FHH extended over eight spans

TABLE 1

Natural frequencies ω (Rad/s) of banded blades for various values of n

n=0	n=1	n=2
168.27474	1003.55780	915.92370
1158.06945	2246.85424	1841.84130
2881.24023	3085.65722	3121.18310
3006.34082	5519.82421	5780.78515
5430.23388	6213.25390	5870.95654

TABLE 2

Comparison of natural frequencies ω (Rad/s) of banded blade problem obtained by the two methods

Natural frequencies obtained by standard Finite element method		Natural frequencies of equivalent rotationally periodic structure					
		n=0		n=1		n=2	
mode no.	ω	mode no.	ω	mode no.	ω	mode no.	ω
1	168.27488	1	168.27478				
2	915.94793					1	915.92370
3	1003.59039			1	1003.55780		
4	1003.59039						
5	1158.11508	2	1158.06945				
6	1842.00316					2	1841.8413
7	2247.14539			2	2246.85424		
8	2247.14395						
9	2881.79444	3	2881.24023				
10	3007.15951	4	3006.34082				
11	3086.54604			3	3085.65722		
12	3086.54604						
13	3122.12299					3	3121.1831

TABLE 3

Natural frequencies ω (Rad/s) of the 8-span equivalent structure for different values of n.

n=0	n=1	n=2	n=3	n=4
1132.9165	1008.7534	780.7320	583.0348	499.7668
1999.0738	2169.4196	2530.0820	2918.3645	3122.9494
6122.3676	5834.2714	5278.9067	4751.3247	4497.9755
7996.6835	8333.3876	9027.6347	9749.6728	10121.1153

TABLE 4

Natural frequencies (ω) of the 2-span beam with three different end conditions.

ω in Rad/s

Beam - HHH	Beam - FHH	Beam - FHF
499.7669	583.3035	780.7315
780.7315	1008.8753	1132.9165
1999.0738	2169.4197	2530.0823
2530.0823	2918.3645	3122.9495
4497.9755	4751.3245	5278.9068

TABLE 5

Comparison of natural frequencies (ω) of HHH beam with those

various cases of n

Beam HHH		n = 0		n = 2		n = 4	
Mode No.	ω (Rad/s)	Mode No.	ω (Rad/s)	Mode No.	ω (Rad/s)	Mode No.	ω (Rad/s)
1	499.7669					1	499.766
2	780.7315			1	780.7320		
3	1999.0738	2	1999.0738				
4	2530.0823			2	2530.0820		
5	4497.9755					3	4497.975
6	5278.9068			3	5278.9067		
7	7996.6836	4	7996.6835				
8	9027.6342			4	9027.6347		

TABLE 6

Comparison of natural frequencies (ω) of FHF beam with those
various cases of n

Beam FHF		n = 0		n = 2		n = 4	
Mode No.	ω (Rad/s)	Mode No.	ω (Rad/s)	Mode No.	ω (Rad/s)	Mode No.	ω (Rad/s)
1	780.7315			1	780.7320		
2	1132.9165	1	1132.9165				
3	2530.0823			2	2530.0820		
4	3122.9495					2	3122.949
5	5278.9068			3	5278.9067		
6	6122.3676	3	6122.3676				
7	9027.6342			4	9027.6347		
8	10121.1155					4	10121.115

TABLE 7

Comparison of natural frequencies (ω) of the Beam FHH with those of various cases of n

Beam FHH		n = 1		n = 3	
Mode No.	ω (Rad/s)	Mode No.	ω (Rad/s)	Mode No.	ω (Rad/s)
1	583.0351			1	583.0348
2	1008.7533	1	1008.7534		
3	2169.4197	2	2169.4196		
4	2918.3645			2	2918.3645
5	4751.3245			3	4751.3247
6	5834.2715	3	5834.2714		
7	8333.3877	4	8333.3876		
8	9749.6726			4	9749.6728

CHAPTER IV

CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

The method of complex constraints is applicable for rotationally periodic structures. This method can be applied to some engineering problems having linear periodicity properties, by appropriate modification of boundary conditions. In the present work this method has been applied to the following two linear periodic structures, by making them linear rotationally periodic:

1. Banded blade structure,
2. Finite continuous beam.

Results have also been obtained by standard finite element method.

Based on the discussion on results presented in chapter III, the following conclusions can be drawn.

i) It is possible to apply the method of complex constraint to linear periodic structures by analysing their equivalent rotationally periodic structures.

ii) The method directly gives orthogonal pairs of eigenvectors for multiple eigenvalues. The two orthogonal vectors can be used to span the 2-D vector subspace.

iii) By this method, it is easier to identify symmetric and antisymmetric modes, as they correspond to some particular values of n.

iv) It is possible to identify a particular eigenvector of the original structure from the modes of it's equivalent rotationally periodic structure by the suggested identification procedure.

v) Since the analysis is reduced to that of a single substructure, the computer memory requirements are much less as compared to the standard finite element method. conversely, for the same memory consumption the method of complex constraints will give more accurate results. The computation time also reduces considerably. The time saving is particularly significant when the number of substructures are large. Transfer matrix method also reduces the matrix size but as it uses an iterative procedure to find natural frequencies, there is not much saving of computation time.

Suggestions for future work

1. It would be interesting to extend this analysis for finding the response of the periodic structures to external excitations.

2. The method can be extended to the analysis of doubly periodic structures.

3. In this method, in order to reduce the the analysis to that of a single substructure, use has been made of the relations between the submatrices of one substructure and the other substructures. For periodic structures these matrices are identical. If the relationship between the submatrices of substructures of almost periodic structures can be obtained then perhaps this method can be extended to the analysis of almost periodic structures also.

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